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OPTIMAL CONTROL OF ARBORESCENT MULTILEVEL INVENTORY - PRODUCTION SYSTEMS

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OPTIMAL CONTROL OF ARBORESCENT MULTILEVEL INVENTORY - PRODUCTION SYSTEMS

CONTROLE OPTIMALE DES CHAINES ARBORESCENTES DE STOCKAGE-PRODUCTION

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ABSTRACT

We present a model for globally optimizing arborescent multilevel inventory-production problems. State-space constraints as well as Poisson distributed demands arriving at any node are considered. For the numerical solution a rate of convergence of order $h \cdot \ln h$ is obtained. We introduce a new algorithm for the fast computation of the numerical solution which allows the resolution of complex problems. Some numerical results are shown. We also study the advantages of cooperative behaviour of nodes versus decentralized optimization.

RESUME

On présente un modèle pour l'optimisation centralisée des chaines multiniveaux de stockage-production. Contraintes sur le space d'états et demandes poissoniennes arrivant à n'importe quel noeud sont considerées. Pour la solution numérique on a obtenu un taux de convergence de l'ordre $h \cdot \ln h$. On présente aussi un nouvel algorithme pour le calcul rapide de la solution numérique qui permet de résoudre problèmes plus généraux. Quelques résultats numériques sont montrés. On étudit les avantages d'une conduite cooperative des noeuds face à une optimisation decentralisée.

CONTENTS

OPTIMAL CONTROL OF ARBORESCENT MULTILEVEL INVENTORY-PRODUCTION SYSTEMS

1. INTRODUCTION

Generally speaking, an arborescent multilevel inventory-production system (AMS) can be defined as a set of installations whose activities are related according to a given hierarchy. In our model, we study such systems for a single product which is distributed through a net with N nodes (each one representing an installation, and with at most one predecessor). Thus we will have a structure which has M different "levels" or sets of nodes having the same hierarchy, but not necessarily the same predecessor. The external supplier will be considered as node 0. We identify the predecessor of i-th node by I(i).

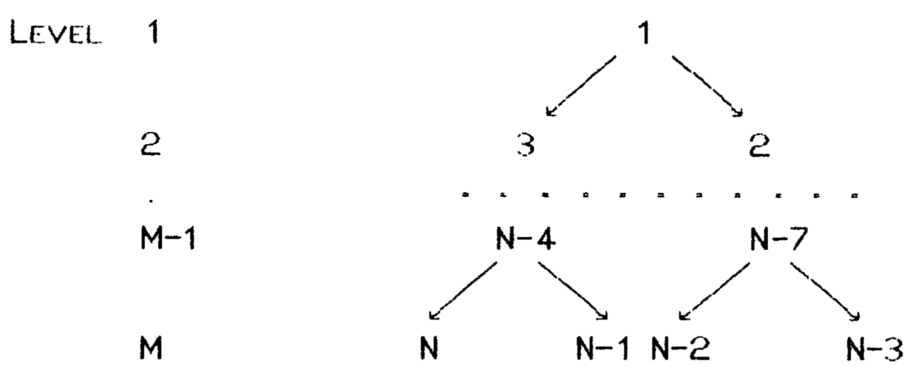


Figure 1

For figure 1 we will have

Node	Predecessor
1	0
2	1
3	1
i	I(i)
....	....
N	N-4

The demand may enter the system at any level of decision. Each node i places orders to its supplier I, for $i=2,...N$, and the exterior supplies the highest installation 1. The purchasing decisions are made at any time and they modify instantaneously the state of the system x, with $x = (x_1, x_2, ..., x_N)$, where x_i represents the amount of stock at installation i. We are interested in the optimal control of stationary AMS with state-space constraints. This means that we consider installations with **finite maximum capacities**, that is, the trajectories of the controlled process stay within a given subset $\Omega \subset \mathbb{R}^N$. Optimization is made with respect to purchasing decisions. Numerical examples of centralized optimization of systems with 4 and 5 nodes are presented in chapter 6. Such chapter ends with many remarks on the cooperative behaviour observed in some subsystems.

2. GENERAL DESCRIPTION OF THE OPTIMAL PROBLEM

2.1. Main features of the model

- The external demand is received at those nodes belonging to the set $\mathcal{J} = \left\{j_1, j_2, \ldots, j_d\right\}$ (*).
- The demand that arrives to each installation has a Poisson distribution, with jump rate λ_j. The jump magnitude is a random variable, $\Delta\xi_j \in \mathbb{R}^+$; its conditional distribution is given by the measure $m_j(\cdot)$ (**).
- Shortage costs are admitted.
- Excess demand is backlogged, up to a maximum amount $|\underline{x}_j|$, $\forall j \in \mathcal{J}$.
- Controls will be orders of impulsive type.
- There is no delay for delivering.
- f – stocking cost
  k – purchasing cost
- At each installation i we know:

(*) For simplicity of notation, in the following we will consider $\mathcal{J} = \{1, 2, \ldots, N\}$.

(**) Although all the results contained in this paper remain valid for a continuous distribution $m_j(\cdot)$, for the sake of simplicity we will restrict our exposition to the case where $m_j(\cdot)$ is concentrated in a finite number of points; i.e. $\Delta\xi_j$ may just take a finite number of values.

x_i^o — initial stock

$\underline{x}_i$ — minimum capacity of node i

($\underline{x}_i < 0$ means unsatisfied demand).

$\overline{x}_i$ — maximum capacity of level i.

Ω — $\prod_{i=1}^{N} [\underline{x}_i, \overline{x}_i]$

2.2 Description of the dynamical system and its control

For j=1,...,N, we assume that between two consecutive orders, each state x_j evolves as a one dimensional piecewise deterministic jump process (strictly speaking, it is a piecewise constant process; we refer to [2], [8], [9] and references therein for a similar setting of the problem). We will denote with σ_t the increasing family of σ-algebras generated by the multidimensional Poisson process.

Let τ_j^{ℓ} be a time when there is a demand arrival for the unidimensional process ξ_j. Our process is defined by: given an initial point $x^o \in \Omega$

$$\left[\begin{array}{l} x_j(0-) = x_j^{\sigma} \\ x_j(\tau_j^{\ell}+) = \mathcal{P}_j(\xi_j(\tau_j^{\ell}+)-\xi_j(\tau_j^{\ell-1}+)\,;x_j(\tau_j^{\ell}-)) \end{array}\right.$$

where

$$\mathcal{P}_j(\gamma_j,z_j) = \begin{cases} z_j-\gamma_j & \text{if } z_j-\gamma_j \in [\underline{x}_j,\bar{x}_j] \\ \underline{x}_j & \text{if } z_j-\gamma_j \in (-\infty\,,\underline{x}_j] \end{cases}$$

Our controls are purchasing orders placed at times θ^{ℓ} which determine transitions $\nu(\theta^{\ell})$ between $x(\theta^{\ell}+)$ and $x(\theta^{\ell}-)$.

$$x(\theta^{\ell}+) = x(\theta^{\ell}-) + \nu(\theta^{\ell})$$

and

θ^{ℓ} is a stopping time adapted to the σ-algebras generated by the ξ process.

$\nu(\theta^{\ell})$ is a random variable $\sigma_{\theta^{\ell}}$-measurable belonging to the set

$$A_x = \left\{ (\nu_1,0,\dots,0)\ /\ \nu_1 \in [0,\bar{x}_1-x_1] \right\} \cup$$

$$\Big\{ (\nu_1,\nu_2,\dots,\nu_N)\ /\ \exists\,!j \text{ with } \nu_j<0 \text{ and } \forall\,\mu\ /\ \nu_\mu>0$$

$$\text{it is } I(\mu)=j,\ \nu_\mu \in [0,\bar{x}_\mu-x_\mu] \text{ and } |\nu_j| = \sum_{\mu/I(\mu)=j} \nu_\mu \le x_j \Big\}$$

and the set of admissible strategies, $\mathcal{A}$, will be the set of adapted policies.

To associate an expected cost to every decision, as we allow the system to receive demands when there is no more stock up to a certain amount $\underline{x}_j < 0$ (for $j \in \mathcal{J}$), we must also consider the effect of accumulating unsatisfied demands as well as the cost $\phi_j(\cdot)$ which will have to be paid when such lower bound is reached, so we will have

$$J(x,\nu(\cdot)) = E\left[\int_0^\infty e^{-\alpha s} f(x(s))\, ds + \sum_{\ell=1}^{\infty} e^{-\alpha\theta^\ell} k(\nu(\theta^\ell)) + \right.$$
$$\left. + \sum_{j=1}^{N} \sum_{\ell=1}^{\infty} e^{-\alpha\tau_j^\ell} \quad \phi_j(x_j(\tau_j^\ell -), \Delta\xi_j^\ell)\right]$$

where $\qquad \Delta\xi_j^\ell = \xi_j(\tau_j^\ell +) - \xi_j(\tau_j^\ell -)$

$$\phi_j(x_j, \Delta\xi_j^\ell) = \begin{cases} \varphi_j(\underline{x}_j - x_j + \Delta\xi_j^\ell) & \text{if } x_j - \Delta\xi_j^\ell < \underline{x}_j \\ 0 & \text{if } x_j - \Delta\xi_j^\ell \geq \underline{x}_j \end{cases}$$

$k : \mathbb{R}^N \longrightarrow \mathbb{R}^+$ represents the ordering cost from installation 1 to the exterior as well as costs related to transfers of products between installations. Besides it holds

$$k(\nu) \geq k_0 > 0 \qquad\qquad \forall \nu$$

and α represents the discount factor.

Therefore the optimal value will be given by

$$V(x) = \inf \left\{ J(x,\nu(\cdot)) \; : \; \nu(\cdot) \in \mathcal{A} \right\}$$

3. SOME PROPERTIES OF THE OPTIMAL COST FUNCTION

3.2 Regularity

In this section we show the optimal cost function is Lipschitz-continuous, provided f and ϕ are also Lipschitz-continuous, with constants L_f and L_ϕ respectively.

Theorem 1:

$$|V(x) - V(y)| \le L_V \; \|x - y\| \qquad \forall\; x, y \in \Omega$$

where
$$L_V = \frac{L_f}{\alpha} + \frac{\Lambda}{\alpha} L_\phi \qquad (*)$$

Proof:

For each $x \in \Omega$, let $\nu(\cdot)$ be an arbitrary control policy and $x(\cdot)$ the corresponding trajectory, for $j=1, \ldots, N$. The trajectory $x(\cdot)$ is a piecewise constant function, with jumps at times τ_j^k and θ^ℓ, with

$$x_j(t) = x_j(0) + \sum_{k=1}^{\infty} \Delta x_j^k \; \chi_{[0,t)}(\tau_j^k) + \sum_{\ell=1}^{\infty} \nu_j(\theta^\ell) \; \chi_{[0,t)}(\theta^\ell)$$

(*) We are considering here $\|\cdot\| = \|\cdot\|_1$ and also for the related inequalities. We will assume our ordering function $k(\cdot)$ satisfies the following property:

Given $I^+ = \left\{ i / \nu_i < 0 \right\}$ and $I^- = \left\{ i / \nu_i \ge 0 \right\}$ it holds

$$|\bar{\nu}_i| \ge |\nu_i| \;\; \forall\; i \in I^+ \text{ implies } \qquad k(\bar{\nu}) \ge k(\nu)$$

where $\quad x_j(0) = x_j$

and, if we define $x_j^k = x_j(\tau_j^k+))$

$$\Delta x_j^k = x_j^k - x_j^{k-1} = \mathcal{P}_j(\xi_j(\tau_j^k)-\xi_j(\tau_j^{k-1}), x_j^{k-1}) \tag{1}$$

Let us now define another admissible process, starting at a given point $y \neq x$:

$$y_j(t) = y_j(0) + \sum_{k=1}^{\infty} \Delta y_j^k \, \chi_{[0,t)}(\tau_j^k) \quad + \sum_{\ell=1}^{\infty} \mu_j(\theta^\ell), \; \chi_{[0,t)}(\theta^\ell)$$

where $\quad y_j(0) = y_j$

and, $\quad y_j^k = y_j(\tau_j^k+))$

$$\Delta y_j^k = y_j^k - y_j^{k-1} = \mathcal{P}_j(\xi_j(\tau_j^k)-\xi_j(\tau_j^{k-1}), y_j^{k-1}) \tag{2}$$

That is, for trajectory $y(\cdot)$, the system receives, if it is possible, as much demand as it does for trajectory $x(\cdot)$; if that is not possible, then it accepts the maximum admissible amount. Concerning controls μ, they are defined by:

$$\mu_j(\theta^\ell) = r^\ell \, \nu_j(\theta^\ell),$$

with

$$r^\ell = \max \left\{ \rho \in [0,1] \; / \; y(\theta^\ell-) + \rho \, \nu(\theta^\ell) \in \Omega \right\}$$

As before, we truncate $\nu(\theta^{\ell})$ so as $y(\cdot)$ stays as near as possible to trajectory $x(\cdot)$.

First, we will prove $\qquad \|x(t)-y(t)\|_1 \le \|x-y\|_1 \qquad \forall\ t. \qquad (3)$

We will split our demonstration in two parts, according to the two different kind of jumps we have, namely those related to τ_j^k and those related to θ^{ℓ}.

a) $$\|x(\tau_j^k+)-y(\tau_j^k+)\|_1 \le \|x(\tau_j^k-)-y(\tau_j^k-)\|_1$$

In fact,

$$\|x(\tau_j^k+)-y(\tau_j^k+)\|_1 = \sum_{i\neq j} |x_i(\tau_j^k+)-y_i(\tau_j^k+)| +$$
$$+ |\mathcal{P}_j(\xi_j(\tau_j^k)-\xi_j(\tau_j^{k-1}),x_j^{k-1}) - \mathcal{P}_j(\xi_j(\tau_j^k)-\xi_j(\tau_j^{k-1}),y_j^{k-1})|$$

As operator $\mathcal{P}_j$ is a projection, it holds

$$|\mathcal{P}_j(\xi_j(\tau_j^k)-\xi_j(\tau_j^{k-1}),x_j^{k-1}) - \mathcal{P}_j(\xi_j(\tau_j^k)-\xi_j(\tau_j^{k-1}),y_j^{k-1})| \le$$

$$\le |(x_j^{k-1}+ \xi_j(\tau_j^k)-\xi_j(\tau_j^{k-1})) - (y_j^{k-1}+ \xi_j(\tau_j^k)-\xi_j(\tau_j^{k-1}))| =$$

$$= |x_j^{k-1} - y_j^{k-1}|.$$

Besides, as $|x_i(\tau_j^k+)-y_i(\tau_j^k+)| = |x_i(\tau_j^k-)-y_i(\tau_j^k-)| \qquad \forall\ i\neq j,$

$$\|x(\tau_j^k+)-y(\tau_j^k+)\|_1 \le \sum_{i\neq j} |x_i(\tau_j^k+)-y_i(\tau_j^k+)| + |x_j^{k-1} - y_j^{k-1}| =$$

$$= \|x(\tau_j^k-)-y(\tau_j^k-)\|_1$$

□

b) $$\|x(\theta^{\ell}+)-y(\theta^{\ell}+)\|_1 \le \|x(\theta^{\ell}-)-y(\theta^{\ell}-)\|_1$$

In fact:

$$x_j(\theta^{\ell}+) - y_j(\theta^{\ell}+) = x_j(\theta^{\ell}-) + \nu_j(\theta^{\ell}) - (y_j(\theta^{\ell}-) + \mu_j(\theta^{\ell})),$$

assume i is a coordinate

where $\mu_i(\theta^{\ell}) = \bar{x}_i - y_i(\theta^{\ell}-)$ and $\nu_i(\theta^{\ell}) \le \bar{x}_i - x_i(\theta^{\ell}-)$

then $\zeta_i = \nu_i(\theta^{\ell}) - \mu_i(\theta^{\ell}) \ge 0$ and $y_i(\theta^{\ell}-) \ge x_i(\theta^{\ell}-)$

therefore $y_i(\theta^{\ell}-) - \zeta_i \ge x_i(\theta^{\ell}-)$

and

$$\begin{aligned} |x_i(\theta^{\ell}+) - y_i(\theta^{\ell}+)| &= |x_i(\theta^{\ell}-) - y_i(\theta^{\ell}-) + \zeta_i| \\ &= - x_i(\theta^{\ell}-) + y_i(\theta^{\ell}-) - \zeta_i \\ &= |x_i(\theta^{\ell}-) - y_i(\theta^{\ell}-)| - \zeta_i \qquad (4) \end{aligned}$$

Let I be the predecessor of i, by definition it holds:

$\mu_I(\theta^{\ell}) = -\mu_i(\theta^{\ell})$ and $\nu_I(\theta^{\ell}) = - \nu_i(\theta^{\ell})$

then

$$\begin{aligned} |x_I(\theta^{\ell}+) - y_I(\theta^{\ell}+)| &= |x_I(\theta^{\ell}-) - y_I(\theta^{\ell}-) - \zeta_i| \\ &\le |x_I(\theta^{\ell}-) - y_I(\theta^{\ell}-)| + \zeta_i \qquad (5) \end{aligned}$$

but

$$\|x(\theta^\ell+)-y(\theta^\ell+)\|_1 = \sum_{\substack{j\neq i\\ j\neq I}} |x_j(\theta^\ell+) - y_j(\theta^\ell+)| + $$

$$+ |x_i(\theta^\ell+) - y_i(\theta^\ell+)| +$$

$$+ |x_I(\theta^\ell+) - y_I(\theta^\ell+)|$$

As $\quad x_j(\theta^\ell+) - y_j(\theta^\ell+) = x_j(\theta^\ell-) - y_j(\theta^\ell-) \quad \forall\ j \neq i,I,$

and by virtue of (4) and (5), we obtain

$$\|x(\theta^\ell+)-y(\theta^\ell+)\|_1 \leq \|x(\theta^\ell-)-y(\theta^\ell-)\|_1$$

The remaining possibilities for $\nu_i(\theta^\ell)$, $\mu_i(\theta^\ell)$ $\nu_I(\theta^\ell)$ and $\mu_I(\theta^\ell)$ may be equally considered to lead to the above result. The result is also valid when we consider simultaneous orders put by installations belonging to the same level L.

□

From **a)** and **b)** it is obvious that (3) holds.

■

Let us now return to $J(y,\mu(\cdot)) - J(x,\nu(\cdot))$. We have:

$$J(y,\mu(\cdot)) - J(x,\nu(\cdot)) \leq \mathbf{E}\Big[\int_0^\infty e^{-\alpha s}\,(f(y(s))-f(x(s)))\,ds +$$

$$+ \sum_{\substack{j=1,N\\ k=1,\infty}} e^{-\alpha\tau_j^k}\,(\phi_j(x_j(\tau_j^k),\Delta\xi_j^k) - \phi_j(y_j(\tau_j^k),\Delta\xi_j^k)) +$$

$$+ \sum_{\ell=1}^{\infty} e^{-\alpha\theta^\ell}(k(\mu(\theta^\ell)) - k(\nu(\theta^\ell)))\Big]$$

but, being $|\mu_I(\theta^\ell)| \leq |\nu_I(\theta^\ell)|$, we have $k(\mu(\theta^\ell)) \leq k(\nu(\theta^\ell))$, that is, $k(\mu(\theta^\ell)) - k(\nu(\theta^\ell)) \leq 0$. Besides, from (3) and the assumptions on f and ϕ, we obtain:

$$J(y,\mu(\cdot)) - J(x,\nu(\cdot)) \leq$$

$$\leq \int_0^\infty e^{-\alpha s} L_f \|y-x\|_1 \, ds + L_\phi \|x-y\|_1 \, E\left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha\tau_j^k}\right]$$

but, it holds that

$$E\left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha\tau_j^k}\right] = \frac{\Lambda}{\alpha}$$

because

$$E\left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha\tau_j^k}\right] = E\left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha\tau_j^k} \left(\chi_{[0,t)}(\tau_j^k) + \chi_{[t,\infty)}(\tau_j^k)\right)\right]$$

$$= E\left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha\tau_j^k} \chi_{[0,t)}(\tau_j^k)\right] +$$

$$+ e^{-\alpha t} E\left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha(\tau_j^k - t)} \chi_{[t,\infty)}(\tau_j^k)\right]$$

by the stationarity of the process, we have

$$E\left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha\tau_j^k}\right] =$$

$$= E\left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha\tau_j^k} \chi_{[0,t)}(\tau_j^1)\right] + e^{-\alpha t} E\left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha\tau_j^k}\right]$$

$$= \sum_{j=1}^{N} \lambda_j \, t + o(t) + e^{-\alpha t} E \left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha \tau_j^k} \right]$$

hence,

$$E \left[\sum_{j=1}^{N} \sum_{k=1}^{\infty} e^{-\alpha \tau_j^k} \right] = \frac{\Lambda \, t}{\alpha \, t} = \frac{\Lambda}{\alpha}$$

□

Consequently,

$$J(y,\mu(\cdot)) - J(x,\nu(\cdot)) \leq \left[\frac{L_f}{\alpha} + \frac{\Lambda}{\alpha} L_\phi \right] \|y-x\|_1$$

but

$$V(y) = \inf\{J(y,\nu)/\nu \in A_y\} \leq J(y,\mu(\cdot))$$

so we have

$$V(y) \leq J(x,\nu(\cdot)) + \left[\frac{L_f}{\alpha} + \frac{\Lambda}{\alpha} L_\phi \right] \|y-x\|_1$$

but, being $\nu(\cdot)$ arbitrary, it holds:

$$V(y) \leq V(x) + \left[\frac{L_f}{\alpha} + \frac{\Lambda}{\alpha} L_\phi \right] \|y-x\|_1$$

Also, as x and y are arbitrary points, we have

$$V(x) \leq V(y) + \left[\frac{L_f}{\alpha} + \frac{\Lambda}{\alpha} L_\phi \right] \|x-y\|_1$$

And the conclusion follows inmediatly.

■

Remark:

Let $g \in \mathscr{V}$, we denote its modulus of continuity with ω_g, i.e.

$$|g(x)-g(y)| \leq \omega_g(\|x-y\|) \qquad \forall\, x,y \in \Omega$$

As we have just shown, V is Lipschitz-continuous provided f and ϕ are Lipschitz-continuous; in the more general case where f and ϕ are merely continuous functions, we can prove (arguing in an identical way) V is continuous and its modulus of continuity satisfies the following bound:

$$\omega_V(\cdot) \leq \frac{1}{\alpha}\left(\omega_f(\cdot) + \Lambda\, \omega_\phi(\cdot)\right)$$

3.2 Formulation of the associated Hamilton-Jacobi-Bellman QVI.

The process defined above is a strong Markov process and the following version of Ito's formula holds, for any $\psi \in C^1(\Omega)$

$$E\left[\psi(x(T))\right] = \psi(x^o) +$$

$$+ E\left[\int_0^T \sum_{j=1}^N \lambda_j \int_0^\infty (\psi(x(t)-\Delta\xi)-\psi(x(t)))\, m_j(d\Delta\xi_j)\, dt\right]$$

Then, by properly using the dynamic programming argument, the optimal cost V must satisfy

$$V(x) = \min \begin{cases} E\left[\int_0^T e^{-\alpha t} f(x(t))dt + \sum_{\substack{j=1,N \\ \ell=1,\infty}} e^{-\alpha\tau_j^\ell} \phi_j(x_j(\tau_j^\ell),\Delta\xi_j^\ell)\, \chi_{[0,T)}(\tau_j^\ell) + e^{-\alpha T} V(x(T)) \right] \\ \min_{\nu\in A_x} \left[k(\nu) + V(x+\nu) \right] \end{cases} \tag{6}$$

The first argument in (6) is related, for $T \downarrow 0$, to the evolution of the process without control, while the second one expresses the ordering phenomenum.

So, similarly to what has been done in [9], (6) leads to

$$V(x) = \min \begin{cases} \dfrac{f(x)}{\alpha} + \displaystyle\sum_{j=1}^{N} \dfrac{\lambda_j}{\alpha} \int [V(\mathcal{P}_j(\Delta\xi_j,x)-V(x)+\phi_j(x,\Delta\xi_j)]\, m_j(d\Delta\xi_j) \\ \min_{\nu\in A_x} \left[k(\nu) + V(x+\nu) \right] \end{cases}$$

As $\int V(x)\, m_j(d\Delta\xi_j) = V(x) \quad \forall j$ and, if we call $\Lambda = \sum \lambda_j$, we can rewrite equation (6):

$$V(x) = \min \begin{cases} \dfrac{f(x)}{\alpha+\Lambda} + \displaystyle\sum_{j=1}^{N} \dfrac{\lambda_j}{\alpha+\Lambda} \int [V(\mathcal{P}_j(x,\Delta\xi_j)+\phi_j(x,\Delta\xi_j)]\, m_j(d\Delta\xi_j) \\ \min_{\nu\in A_x} \left[k(\nu) + V(x+\nu) \right] \end{cases} \tag{7}$$

3.3 Associated Quasi-Variational Inequalities. Properties

We will study here the QVI defined in (7), the existence and uniqueness of its solution as well as the functional space where we look for it. Also, we consider the constructive algorithm giving the solution and its rate of convergence; here we find a similar bound to that obtained by Hanouzet-Joly in [6]. We only ask the function $k(\cdot)$ to have a positive lower bound, that is

$$k(\cdot) \geq k_o > 0.$$

and the condition

$$f,\ \phi \in L^{\infty}(\Omega), \qquad \alpha > 0$$

3.3.1.- Solution of the QVI via a constructive method.

We introduce here an operator $\mathcal{M} : C(\Omega) \longrightarrow C(\Omega)$ such that for each $x \in \Omega$ it is

$$\mathcal{M}u(x) = \min_{\nu \in A_x} \left\{ k(\nu) + u(x+\nu) \right\}$$

We will shorten $\mathcal{V} = C(\Omega)$.

We also define the application σ: $\mathcal{V} \longrightarrow \mathcal{V}$ such that for every ψ in $\mathcal{V}$, $\sigma(\psi)$ solves the following problem:

Find the maximum solution of the following VI:

$$\begin{cases} v \le \beta\, I(v,\phi,f) \\ v \le \psi \end{cases} \tag{8}$$

here we have defined $\beta = \dfrac{1}{\alpha+\Lambda}$ and $I(v,\phi,f)$ as the following linear operator:

$$(I(v,\phi,f))(x) = f(x) + \sum_{j=1}^{N} \lambda_j \int [v(\mathcal{P}_j(\Delta\xi_j,x)+\phi_j(x,\Delta\xi_j)]\ m_j(d\Delta\xi_j)$$

The problem of finding the solution of (7) is equivalent to find the fixed point of operator $M = \sigma\circ\mathcal{M}$, that is why in the following sections we will show some properties of operators σ, $\mathcal{M}$ and their composition.

This fixed point can be iteratively computed by the Bensoussan-Lions algorithm ([3]):

Step 0: Give $v^{o} \in \mathcal{V}$ and set $m = 0$.

Step 1: Define $v^{m+1} = M(v^{m})$

Step 2: Set $m = m+1$, and go to Step 1.

We will show $v^{m} \longrightarrow V$, moreover, we will prove the following geometrical rate of convergence holds (for $\delta(v^{0}) \in (0,1]$):

$$\|v^{m} - V\|_{C(\Omega)} \le K(v^{0})\ (1-\delta(v^{0}))^{m}$$

3.3.2. Operator σ. Computation and properties.

We can state:

Theorem 2:

i) There exists a maximum element $\sigma(\psi)$ of the set of functions verifying (8).

ii) $\sigma(\psi) = \lim_{m \to \infty} W_\psi^m(u)$, where, for $u \in \mathscr{V}$ arbitrary,

$$W_\psi(u) = \min(\psi, \beta I(u, \phi, f)) \qquad (9$$

iii) σ is an increasing operator.

iv) If we define a *subsolution* of (8), u, when $u \le W_\psi(u)$ then we have $\sigma(\psi) \ge u$ $\forall u$ subsolution (i.e., $\sigma(\psi)$ is the maximum subsolution).

v) If we define a *supersolution* of (8), u, when $u \ge W_\psi(u)$; then we have $\sigma(\psi) \le u$ $\forall u$ supersolution (i.e., $\sigma(\psi)$ is the minimum supersolution).

Proof:

i), *ii)* and *iv)*: First we show $W_\psi(\cdot)$ is a contractive application, that is,

$$\|W_\psi(u) - W_\psi(v)\|_{C(\Omega)} \le \beta\, \Lambda\, \|u - v\|_{C(\Omega)}$$

for every $u, v \in \mathscr{V}$, with $\beta\Lambda < 1$.

- If $W_\psi(v) = \psi$, then $(W_\psi(u)-W_\psi(v))(x) \le \psi(x)-\psi(x) = 0 \;\forall x\in\Omega$.

- If $W_\psi(v) = \beta I(v,\phi,f)$, then $W_\psi(u)-W_\psi(v) \le \beta I(u,\phi,f)-\beta I(v,\phi,f) \le$

$$\le \beta \sum_{j=1}^{N} \lambda_j \int [u(\mathcal{P}_j(x,\Delta\xi_j))-v(\mathcal{P}_j(x,\Delta\xi_j))]\; m_j(d\Delta\xi_j)$$

then

$$W_\psi(u)-W_\psi(v) \le \beta\ \Lambda\ \|u-v\|_{C(\Omega)}$$

the similar inequality for $W_\psi(v)-W_\psi(u)$ can be shown in a completely simmetrical way, and we obtain

$$\|W_\psi(u)-W_\psi(v)\|_{C(\Omega)} \le \beta\ \Lambda\ \|u-v\|_{C(\Omega)}$$

and, as $\beta\Lambda < 1$, we conclude $W_\psi(\cdot)$ is a contraction in $\mathcal{V}$. □

Since $W_\psi(\cdot)$ is contractive, we know there exists a limit for the iterative algorithm which is the fixed point of W_ψ, i.e.

$$\exists\ \sigma(\psi) \in \mathcal{V} \ /\ \sigma(\psi) = W_\psi(\sigma(\psi)) = \lim_{m\to\infty} W_\psi^m(u) \qquad \forall\, u \in \mathcal{V}$$

We will see $W_\psi : \mathcal{V} \longrightarrow \mathcal{V}$ is an increasing application:

Let $u, v \in \mathcal{V}$ such that $u \le v$ then:

- If $(W_\psi(v))(x) = \psi(x)$, then, by (9) it is

$$(W_\psi(u))(x) \le (W_\psi(v))(x) \qquad \forall x\in\Omega.$$

- If $(W_\psi(v))(x) = \beta(I(v,\phi,f))(x)$, then

$(W_\psi(u)-W_\psi(v))(x) \le \beta(I(u,\phi,f)-\beta I(v,\phi,f))(x) \le$

$$\leq \beta \sum_{j=1}^{N} \lambda_j \int [u(\mathcal{P}_j(x,\Delta\xi_j)-v(\mathcal{P}_j(x,\Delta\xi_j)] \, m_j(d\Delta\xi_j) \leq 0$$

□

Now let u be a subsolution of (8), then $u \leq W_\psi(u)$ but as $W_\psi(\cdot)$ is an increasing operator, we can affirm

$$W_\psi(u) \leq W_\psi^2(u),$$

and inductively,

$$u \leq W_\psi^m(u) \leq W_\psi^{m+1}(u)$$

so in the limit we will have

$$u \leq \lim_{m\to\infty} W_\psi^m(u) = \sigma(\psi)$$

iii) We see first $W_\psi(\cdot)$ is an increasing application in ψ. Let $\psi_1 \leq \psi_2$, then $W_{\psi_1}(u) = \min(\psi_1, \beta l(u,\phi,f)) \leq \min(\psi_2, \beta l(u,\phi,f)) = W_{\psi_2}(u)$

Therefore, $\sigma(\psi_1) = \lim_{m\to\infty} W_{\psi_1}^m(u) \leq \lim_{m\to\infty} W_{\psi_2}^m(u) = \sigma(\psi_2)$.

v) It can be shown in an completely similar way as it was done in iv), mutatis mutandis.

3.3.3 The operator $M = \sigma \circ \mathcal{M}$. Properties.

We will show at this section there is a unique fixed point of operator M, and that this fixed point is the solution of (7). Reciprocally, every solution of (7) is a fixed point of M. Also we will state some properties of the operator considered which will be of special interest when proving the rate of convergence of the algorithm of Bensoussan-Lions.

Theorem 3:

i) M is an increasing operator.

ii) M is concave: $\forall\ u, v \in \mathcal{V}$, $\theta \in [0,1]$, it holds

$$\theta\, M(u) + (1-\theta)\, M(v) \le M(\theta u + (1-\theta) v)$$

iii) $\forall\ u \in \mathcal{V}$, $\exists\ \underline{v}$ and $\bar{v}$ in $\mathcal{V}$ such that

$$\underline{v} \le u \le \bar{v}$$

and $\qquad \exists\ \delta \in (0,1]$ such that $\underline{v} + \delta(\bar{v} - \underline{v}) \le M(\underline{v})$, $\qquad$ (10)

iv) $\forall\ v \le \bar{v}$ it is $M(v) \le \bar{v}$.

Proof:

i) Let v_1, $v_2 \in \mathcal{V}$ such that $v_1 \le v_2$, it is clear $\mathcal{M}(v_1) \le \mathcal{M}(v_2)$, and as $\sigma(\cdot)$ is also an increasing operator, M will be increasing.

ii) First we will show $\mathcal{M}(\cdot)$ is a concave function: let $u,v \in \mathcal{V}$ and $\theta \in [0,1]$. We have:

$$(\mathcal{M}(\theta u+(1-\theta)v))(x) = \min_{\nu\in A_x} \left[k(\nu) + (\theta u+(1-\theta)v)(x+\nu) \right]$$

$$= k(\bar{\nu})+(\theta u+(1-\theta)v)(x+\bar{\nu}) = k(\bar{\nu})+\theta u(x+\bar{\nu})+(1-\theta)v(x+\bar{\nu})$$

$$= \theta\ (\ k(\bar{\nu}) + u(x+\bar{\nu})\ ) + (1-\theta)\ (\ k(\bar{\nu}) + v(x+\bar{\nu})\ )$$

$$\geq \theta\ (\mathcal{M}(u))(x) + (1-\theta)\ (\mathcal{M}(v))(x)$$

From its definition, we have:

- $M(u) \leq \mathcal{M}(u)$, and $M(v) \leq \mathcal{M}(v)$, so if we do the convex sum of $M(u)$ and $M(v)$, then we will obtain: .

$$\theta\ M(u) + (1-\theta)\ M(v) \leq \theta\ \mathcal{M}(u) + (1-\theta)\ \mathcal{M}(v) \leq \mathcal{M}(\theta u + (1-\theta)v) \quad (11)$$

- $M(u) \leq \beta I(M(u),\phi,f)$, and $M(v) \leq \beta I(M(v),\phi,f)$, and the convex sum will verify:

$$\theta\ M(u) + (1-\theta)\ M(v) \leq \theta\ \beta\ I(M(u),\phi,f) + (1-\theta)\ \beta\ I(M(v),\phi,f) \quad (12)$$

but operator $I(\cdot,\cdot,\cdot)$ is linear, so

$$\theta\ \beta\ I(M(u),\phi,f) + (1-\theta)\ \beta\ I(M(v),\phi,f) =$$

$$= \beta\ I(\theta M(u)+(1-\theta)M(v),\phi,f) \quad (13)$$

By virtue of (11)-(13), $\theta M(u)+(1-\theta)M(v)$ is a subsolution of (4).

setting $\psi = \mathcal{M}(\theta u + (1-\theta)v)$ so, by Theorem 2, we know

$$\theta M(u)+(1-\theta)M(v) \le \sigma(\psi) = \sigma\circ\mathcal{M}(\theta u + (1-\theta)v)) = M(\theta u+(1-\theta)v)$$

iii) Given $u \in \mathcal{V}$, we know $-\|u\|_{C(\Omega)} \le u \le \|u\|_{C(\Omega)}$,

so let $$K = \max\left\{ \|u\|_{C(\Omega)} , \frac{\beta\ \Lambda}{1-\beta\Lambda} . \frac{(\|f\| + \Lambda\|\phi\|)}{2} \right\}$$

then, defining $\underline{v} \equiv -K$ and $\bar{v} = K$, clearly $\underline{v}$ and $\bar{v} \in \mathcal{V}$ and

also $$\underline{v} \le u \le \bar{v} .$$

Let us now show (10) is true for δ defined by

$$\delta = \min\left\{ 1, \frac{k_o}{2K} , \frac{1}{2} - \frac{\beta\ \Lambda}{1-\beta\Lambda} \frac{\|f\| + \Lambda\|\phi\|}{4\ K} \right\} \tag{14}$$

$$\underline{v} + \delta(\bar{v}-\underline{v}) = -K + \delta\ 2K \le -K + \frac{k_o}{2K} 2K = -K + k_o$$

$$\le k(\nu) + \underline{v}(x+\nu) \qquad \forall\ x \in \Omega,\ \forall\ \nu \in A_x,$$

in particular, for $\underline{\nu}$ such that

$$k(\underline{\nu}) + \underline{v}(x+\underline{\nu}) = \min_{\nu\in A_x}\left[k(\nu) + \underline{v}(x+\nu) \right] = (\mathcal{M}(\underline{v}))(x)$$

so

$$\underline{v} + \delta(\bar{v}-\underline{v}) \le \mathcal{M}(\underline{v}) .$$

Also,

$$\underline{v} + \delta(\bar{v}-\underline{v}) = (2\delta - 1)\ K \tag{15}$$

On the other side, it can be easily seen that

$$\beta\, I(\underline{v} + \delta(\bar{v}-\underline{v}),\phi,f) \geq \beta\, \Lambda\, (\, \underline{v} + \delta(\bar{v}-\underline{v})\,) - \beta\, \Lambda\, \|\phi\| - \beta\, \|f\| =$$

$$= \beta\, \Lambda\, (\, K\, (2\delta-1)\,) - \beta\, \Lambda\, \|\phi\| - \beta\, \|f\| \tag{16}$$

By virtue of (14), (15) and (16),

$$\underline{v} + \delta(\bar{v}-\underline{v}) \leq \beta\, I(\underline{v} + \delta(\bar{v}-\underline{v}),\phi,f)$$

Therefore $\underline{v} + \delta(\bar{v}-\underline{v})$ is a subsolution of (8), with $\psi = \mathcal{M}(\underline{v})$, and

it holds $\underline{v} + \delta(\bar{v}-\underline{v}) \leq M(\underline{v})$.

iv) Let $v \in \mathcal{V}$ such that $v \leq \bar{\bar{v}}$, we know $M(v) \leq M(\bar{\bar{v}})$, but we have

$$M(\bar{v}) = \lim_{m\to\infty} W^m_{\mathcal{M}\bar{v}}(u) \qquad\qquad \forall\, u \in \mathcal{V}$$

in particular, for $u = \bar{\bar{v}}$.

As $\mathcal{M}(\bar{v}) = k_o + \bar{v}$, it holds $W_{\mathcal{M}(\bar{\bar{v}})}(\bar{v}) = \beta\, I(\bar{v},\phi,f) \leq \bar{v}$ by its

definition.

So $$M(\bar{v}) = \lim_{m\to\infty} W_{\mathcal{M}(\bar{\bar{v}})}(\bar{v}) \leq \bar{v}$$

3.3.4 Convergence of the Bensoussan-Lions algorithm

Theorem 4:

$$\forall\, u \in \mathcal{V} \quad \text{it holds} \quad \|M^m(u) - V\|_{L(\Omega)} \leq k(u)\, (1-\delta(u))^m$$

Proof:

Let $v, w \in \mathcal{V}$ such that $\underline{v} \leq v, w \leq \bar{v}$, we know there exist $\theta, \tau \in [0,1]$ such that

$$\tau(\underline{v}-w) \leq v-w \leq \theta\ (v-\underline{v})$$

Let $z = (1-\theta)\ v + \theta\ \underline{v}$, then $z \leq w$, and , by the concavity of $M(\cdot)$ it holds

$$M(w) \geq M(z) \geq (1-\theta)\ M(v) + \theta\ M(\underline{v})$$

but

$$(1-\theta)\ M(v) + \theta\ M(\underline{v}) = M(v) - \theta\ (M(v)-M(\underline{v}))$$

so

$$M(w) \geq M(v) - \theta\ (M(v)-M(\underline{v}))$$

Let $u(\delta) = \delta\bar{v} + (1-\delta)\underline{v}$, because of (10), we know $u(\delta) \leq M(\underline{v})$, then

$$M(w) \geq M(v) - \theta\ (M(v)-u(\delta))$$

that is,

$$M(w) \geq M(v) - \theta\ (M(v) - \delta\bar{v} - (1-\delta)\underline{v})$$

but being $v \leq \bar{v}$, it is $M(v) \leq \bar{v}$

$$M(w) \geq M(v) - \theta\ (M(v) - \delta M(v) - (1-\delta)\underline{v})$$

hence,

$$M(w) \geq (1-\theta(1-\delta))M(v) - (1-\delta)\underline{v}$$

or equivalently,

$$M(v) - M(w) \leq \theta\ (1-\delta)(M(v)-\underline{v})$$

Interchanging v and w, we can also obtain

$$\tau\ (1-\delta)\ (\underline{v}-M(w)) \leq M(v) - M(w)$$

That is, it holds

$$\tau\ (1-\delta)(\underline{v}-M(w)) \leq M(v) - M(w) \leq (1-\delta)\ \theta\ (M(v)-\underline{v})$$

Now, if we set $w = u$, and $v = V$, as $u^1 = M(u)$, we will have:

$$\tau\ (1-\delta)\ (\underline{v}-u^1) \leq V - u^1 \leq (1-\delta)\ \theta\ (V-\underline{v})$$

And, by induction

$$\tau\ (1-\delta)^m\ (\underline{v}-u^m) \leq V - u^m \leq (1-\delta)^m\ \theta\ (V-\underline{v})$$

From the last inequalities, we obtain the bound we were looking for, by setting $K(u) = 2\ K \max\ (\tau\ ,\ \theta)$

4. NUMERICAL SOLUTION

Let us set the following system of inequalities:

$$w(x) \le \begin{cases} \dfrac{f(x)}{\alpha+\Lambda} + \displaystyle\sum_{j=1}^{N} \dfrac{\lambda_j}{\alpha+\Lambda} \int [w(F_j(x,q_j)+\Phi_j(x,q_j)]\ m_j(dq_j) \\ \displaystyle\min_{\nu\in\mathcal{A}_x} \left[\ k(\nu) + w\ (x+\nu)\ \right] \end{cases} \tag{17}$$

We know $V \in \mathcal{V}$ is the maximum solution of (17). To solve the problem numerically, we approximate this unknown value function with a maximum solution of a direct discretization of (17).

4.1. The discretized problem

Let us assume we have the following discretization pattern:

- Ω is approximated by

$$\Omega_h = \prod_{i=1}^{N} \left\{ \underline{x}_i + j.h_i\ ,\ j = 0,\dots,N_i-1 \right\}$$

Here N_i represents the number of points chosen to discretize the stock at the i-th installation,

$$\text{and } h_i = \frac{\bar{x}_i - \underline{x}_i}{N_i - 1} \qquad i=1,2,\dots,N.$$

Given a set of parameters Nq_i, we define

$$H_i = \frac{\bar{x}_i - \underline{x}_i}{Nq_i - 1} \qquad i=1,2,\dots,N.$$

and we set $$h = \max \left\{ h_i , H_i \ / \ i=1,2,\ldots,N \right\}.$$

- As the probability distribution $m_j(\cdot)$ is discrete, we have:

$$\int g(\Delta\xi_j)\ m_j(d\Delta\xi_j) = \sum_{\jmath\in\vartheta_j} g(\Delta\xi_{\jmath j})\ m_{\jmath j}$$

being $\left\{ (m_{\jmath j}, \Delta\xi_{\jmath j}) \right\}_{\jmath\in\vartheta_j}$ a finite family such that $\sum_{\jmath\in\vartheta_j} m_{\jmath j} = 1.$

We will also need to discriminate two subsets of indexes, which we will denote by $\vartheta^-(x_j)$ (and $\vartheta^+(x_j)$), representing the sets of indexes $\jmath$ in ϑ_j such that, for a given value of x_j, it holds

$$x_j - \underline{x}_j < \Delta m_{\jmath j} \qquad \text{(or } x_j - \underline{x}_j \geq \Delta\xi_{\jmath j}\text{, respectively).}$$

- Discretization of the set of functions $\mathscr{W}$ will yield

$$\mathscr{W}_h = \left\{ w : \Omega_h \longrightarrow \mathbb{R} \ / \ w \text{ is a finite element of type } Q^1 \right\}$$

Now, let $x^k = (x_1^k, x_2^k, \ldots, x_N^k)$ be a node in Ω_h, with $k=1,\ldots,N_h$, being $N_h = N_1 \cdot N_2 \cdot \ldots \cdot N_N$.

The discretization of the free evolution equation gives, for every w in $\mathscr{W}_h$:

$$w(x^k) \leq \frac{1}{\alpha + \Lambda} \left\{ f(x^k) + \sum_{j=1}^{N} \lambda_j * \right.$$

$$\left. \left[\sum_{j \in \vartheta^+} w(x^k - \Delta\xi_{jj} e_j)\, m_{jj} + \sum_{j \in \vartheta^-} \left[w(x^k + (\underline{x}_j - x_j^k) e_j) + \varphi_j(\underline{x}_j - x_j^k + \Delta\xi_{jj}) \right] m_{jj} \right] \right\}$$

If $x^k - \Delta\xi_{jj} e_j \notin \Omega_h$ we perform linear interpolation on w (just in one coordinate).

Similarly, the discretized equation for the purchasing phenomenum gives

$$w(x^k) \leq \mathcal{M}_h(w)(x^k)$$

$$\mathcal{M}_h(w)(x^k) = \left[\begin{array}{l} \min \left\{ k(q) + w(x^k + q - \sum_i q_i e_I) \quad / \ \forall\ i \neq 1 \right. \\ \qquad \left. q = \sum_i q_i e_i \text{, with } q_i \in Q_i(x^k) \text{ and } \sum_i q_i \leq x_I^k \right\} \\ \\ \min \left\{ k(q) + w(x^k + q_1 e_1) \quad / \quad q_1 \in Q_1(x^k) \right\} \end{array} \right.$$

with

$$Q_i(x^k) = \left\{ m_i . H_i \ / \ m_i = 0, \ldots, \left[\frac{x_i^k - \underline{x}_i}{H_i} \right] \right\}$$

(here [y] represents the nearest integer less or equal than y)

• We define the direct discretization of (17) by:

$$w(x^k) \leq \begin{cases} \dfrac{1}{\alpha + \Lambda} \Big\{ f(x^k) + \sum_{j=1}^{N} \lambda_j * \Big[\sum_{i\in\vartheta^+} w(x^k - \Delta\xi_{ij} e_j)\, m_{ij} + \\ \quad + \sum_{i\in\vartheta^-} \Big[w(x^k + (\underline{x}_j - x_j^k) e_j) + \varphi_j(\underline{x}_j + \Delta\xi_{ij} - x_j^k)\Big] m_{ij} \Big]\Big\} \\ \mathcal{M}_h(w)(x^k) \end{cases} \quad (QVI_h$$

Similarly as we have done in -3.1, we will shorten $\mathcal{W} = \mathcal{W}_h$. We define the application σ_h: $\mathcal{W} \longrightarrow \mathcal{W}$ such that for every ψ_h in $\mathcal{W}$, $\sigma_h(\psi_h)$ solves the following problem:

$$\left[\ \text{Find the maximum solution of the following VI:} \quad \begin{cases} v_h \leq \beta\, I_h(v_h, \Phi, f) \\ v_h \leq \psi_h \end{cases}\right. \quad (18)$$

where by $I_h(\cdot,\cdot,\cdot)$ we mean the following discrete operator (defined in $\mathcal{W}_h$):

$$(I_h(w,\phi,f))(x^k) = \Big\{ f(x^k) + \sum_{j=1}^{N} \lambda_j *$$

$$\Big[\sum_{i\in\vartheta^+} w(x^k - \Delta\xi_{ij} e_j)\, m_{ij} + \sum_{i\in\vartheta^-} \Big[w(x^k + (\underline{x}_j - x_j^k) e_j) + \varphi_j(\underline{x}_j - x_j^k + \Delta\xi_{ij})\Big] m_{ij} \Big] \Big\}$$

As we have shown in **3.1.2**, we can prove the following

Theorem 5:

i) There exists a maximum element of the set of functions verifying (18).

ii) $\sigma_h(\psi) = \lim_{m \to \infty} W^m_{h\psi}(u_h)$, where $W_{h\psi}(u_h) = \min(\psi, \beta I_h(u_h; \phi, f$

for $u_h \in \mathcal{W}$ arbitrary.

iii) σ_h is an increasing operator.

iv) If we define a *subsolution* of (18), u_h, when $u_h \leq W_{h\psi}(u_h)$; then we have $\sigma_h(\psi) \geq u_h$ $\forall u_h$ subsolution (i.e., $\sigma_h(\psi)$ is the maximum subsolution).

v) If we define a *supersolution* of (18), u_h, when $u_h \geq W_{h\psi}(u_h)$; then we have $\sigma_h(\psi) \leq u_h$ $\forall u_h$ supersolution (i.e., $\sigma_h(\psi)$ is the minimum supersolution).

Our discretized problem will be:

$$\boxed{\text{To find } V_h, \text{ the maximum subsolution of } (\mathrm{QVI}_h)}$$

The problem of finding the maximum subsolution of (QVI_h) is equivalent to find the fixed point of operator $M_h = \sigma_h \circ \mathcal{M}_h$. In fact, arguing as we have done for the continuous case , but working now in the subspace $\mathcal{W}_h$ of $C(\Omega)$, we could show there is a unique fixed point of operator M_h, and that this fixed point is the maximum subsolution of (QVI_h). Reciprocally, the maximum subsolution of (QVI_h) is a fixed point of M_h. Writting down our results in a more formal way, we have:

Theorem 6:

i) M_h is an increasing operator.

ii) M_h is concave: $\forall\ u_h, v_h \in \mathcal{W}$, $\theta \in [0,1]$, it holds

$$\theta\ M_h(u_h) + (1-\theta)\ M_h(v_h) \leq M_h(\theta u_h + (1-\theta) v_h)$$

iii) $\forall\ u_h \in \mathcal{W}$, $\exists\ \underline{v}_h$, $\bar{v}_h \in \mathcal{W}$ such that $\underline{v}_h \leq u_h \leq \bar{v}_h$ and

$$\exists\ \delta_h \in (0,1] \text{ such that } \underline{v}_h + \delta_h(\bar{v}_h - \underline{v}_h) \leq M_h(\underline{v}_h) \qquad (14)$$

iv) $\forall\ v_h \leq \bar{v}_h$ it is $M_h(v_h) \leq \bar{v}_h$.

As is **3.4.** we can state the following:

Theorem 7:

The fixed point of operator M_h can be iteratively computed by the discrete Bensoussan-Lions algorithm:

Step 0: Give $v_h^{\circ} \in \mathscr{W}$ and set $m = 0$.

Step 1: Define $v_h^{m+1} = M_h(v_h^m)$

Step 2: Set $m = m+1$, and go to Step 1.

Moreover, $\forall\, v_h^{\circ} \in \mathscr{W}$ it holds the following geometrical rate of convergence:

$$\|v_h^m - V_h\| = \|M_h^m(v_h^{\circ}) - V_h\| \le K(v_h^{\circ})\ (1-\delta_h(v_h^{\circ}))^m \tag{19}$$

4.2 Rate of Convergence

We will prove $\|V-V_h\|_{C(\Omega)} = \mathcal{O}(h\cdot \ln h)$.

As we are working with $\mathscr{V} = C(\Omega)$ and $\mathscr{W}_h$ is a subspace of it, for better legibility, we will write $\|\cdot\|$ instead of $\|\cdot\|_{C(\Omega)}$.

Clearly,

$$\|V-V_h\| \le \|V-M^m(u)\| + \|M^m(u)-M_h^m(u_h)\| + \|M_h^m(u_h)-V_h\| \tag{20}$$

Up to now, we obtained an upper bound for the approximation of V by the solution of succesive stopping time problems (Bensoussan-Lions algorithm) , where in each one we only admit a fixed amount of controls (just m impulses). This bound is expressed in terms of the first element of the sequence and it is valid provided there exists $k_o>0$, a positive lower bound for the ordering costs and data f and ϕ belong to $\mathcal{V}$. That is, we know

$$\|V-M^m(u)\| \le K(u) \ (1-\delta(u))^m$$

Similarly we have the following bound for the convergence of the solutions of the discretized stopping time problem :

$$\|M_h^m(u_h)-V_h\| \le K(u_h) \ (1-\delta_h(u_h))^m$$

It only remains us to state a bound for the convergence of the discretized stopping time problem to the continuous one. That is, we will show in **4.2.1** that

$$\|M^m(u)-M_h^m(u_h)\| \le \left[m \ L_k + (m + \frac{\Lambda}{\alpha}) \ L_v \right] h$$

Consequently, (20) gives

$$\|V-V_h\| \le K(u) \ (1-\delta(u))^m + \left[m \ L_k + (m + \frac{\Lambda}{\alpha}) \ L_v \right] h +$$

$$+ K(u_h)(1-\delta_h(u_h))^m \qquad (21)$$

and we obtain

$$\|V-V_h\| \leq C_h\ h\ \ln\left[\ D_h/h\ \right]$$

where $C_h = -\dfrac{L_k + L_v}{\ln \Psi}$ and $D_h = \dfrac{\Psi K}{C_h}\ e^{\Lambda L_v/\alpha C_h}$

With Ψ and K to be defined below.

4.2.1 Convergence of the discretized stopping time problem to the continuous problem. Computation of D_h and C_h.

We will assume here $k(\cdot)$ is a Lipschitz-continuous function with Lipschitz constant L_k.

Let $m \geq 1$, we have

$$\begin{cases} u^m \leq \beta\ I(u^m,\phi,f) \\ u^m \leq \mathcal{M}u^{m-1} \end{cases}$$

$$\begin{cases} u_h^m \leq \beta\ I_h(u_h^m,\phi,f) \\ u_h^m \leq \mathcal{M}_h u_h^{m-1} \end{cases}$$

We start our iterations from $u^o = \beta\ I(u^o,\phi,f)$

and $$u_h^o = \beta \; I_h(u_h^o,\phi,f)$$

We are looking for un upper bound for $\varepsilon^m = \|M^m(u)-M_h^m(u_h)\|$.

Let $m = 0$, we have

$$u^o(x)-u_h^o(x) = \frac{1}{\alpha+\Lambda}\left[\sum_j \lambda_j \int [u^o(\mathcal{P}_j(\Delta\xi_j,x_j)-u_h^o(\mathcal{P}_j(\Delta\xi_j,x_j)]m_j(d\Delta\xi_j)\right]$$

As $u_h^o \in \mathcal{W}_h$, we have

$$u_h^o(\mathcal{P}_j(\Delta\xi_j,x_j)) = \sum_k \mu_{kj}u_h^o(x_j^k) \qquad \text{with } \sum_k \mu_{kj} = 1 \quad \forall j \tag{22}$$

$$u^o(x)-u_h^o(x) = \frac{1}{\alpha+\Lambda}\left[\sum_j \lambda_j \int [u^o(\mathcal{P}_j(\Delta\xi_j,x_j)-\sum_k \mu_{kj}u_h^o(x_j^k)m_j(d\Delta\xi_j)\right]$$

but, we can rewrite

$$u^o(\mathcal{P}_j(\Delta\xi_j,x_j)) - \sum_k \mu_{kj}u_h^o(x_j^k) = u^o(\mathcal{P}_j((\Delta\xi_j,x_j) - \sum_k \mu_{kj}u^o(x_j^k) +$$

$$+ \sum_k \mu_{kj}(u^o(x_j^k)-u_h^o(x_j^k))$$

As we have done when proving V is Lipschitz-continuous, it could be shown u^m are also Lipschitz-continuous, with the same Lipschitz constant as V; thus as

$$|\mathcal{P}_j(\Delta\xi_j,x_j)-x_j^k| \le h \;\; \forall \; k,j,$$

taking absolute value in both members, we obtain

$$\varepsilon^{o} \le \frac{\Lambda}{\alpha + \Lambda} \left[L_V \, h + \varepsilon^{o} \right]$$

that is,

$$\varepsilon^{o} \le \frac{\Lambda}{\alpha} L_V \, h$$

Let us now set $m = 1$. When $u_h^{1}(x)$ reachs the obstacle $\mathcal{M}_h u_h^{o}$, we will have

$$u_h^{1}(x) = k(\pi) + u_h^{o}(x+\pi), \qquad \text{for } \pi \in A_x$$

then

$$u^{1}(x)-u_h^{1}(x) \le u^{1}(x) - (k(\pi) + u_h^{o}(x+\pi))$$

$$\le k(\pi)+u^{o}(x+\pi) - k(\pi)-u_h^{o}(x+\pi) = u^{o}(x+\pi)-u_h^{o}(x+\pi)$$

but, as in (22), we know $u_h^{o}(x+\pi) = \sum_k \mu_k \, u_h^{o}(x^k)$ with $\sum_k \mu_k = 1$, so we will have, doing a similar argument as before, that

$$u^{o}(x+\pi)-u_h^{o}(x+\pi) \le L_V \, h + \sum_k \mu_k \, (u_h^{o}(x^k) - u_h(x^k))$$

hence

$$\varepsilon^{1} \le L_V \, h + \varepsilon^{o}$$

On the other side, when we consider $u_h^{1}(x)-u^{1}(x)$, and $u^{1}(x)$ reachs the obstacle $\mathcal{M}u^{o}(x)$, we have:

$$u_h^{1}(x) - u^{1}(x) \le u_h^{1}(x) - (k(\pi) + u^{o}(x+\pi)) \qquad \pi \in A_x$$

The construction of our discrete procedure allows us to assure there is a discrete control π_h such that

$$\|\pi-\pi_h\| \leq h$$

so

$$|k(\pi) + u^o(x+\pi) - k(\pi_h) - u^o(x+\pi_h)| \leq (L_k+L_V) \|\pi-\pi_h\| \leq (L_k+L_V) h$$

Therefore

$$|u_h^1(x) - u^1(x)| \leq |u^o(x+\pi_h) - u_h^o(x+\pi_h)| + (L_k+L_V) h$$

that is,

$$\varepsilon^1 \leq \varepsilon^o + (L_k+L_V) h$$

Summing up,

$$\varepsilon^1 \leq \max \left[\frac{\Lambda}{\alpha} L_V h , \varepsilon^o + (L_k+L_V) h \right]$$

and inductively,

$$\varepsilon^m \leq \frac{\Lambda}{\alpha} L_V h + m (L_k+L_V) h \quad \blacktriangle$$

Computation of C_h and D_h.

If we define now

$$\Psi = \max \left[(1-\delta), (1-\delta_h) \right] \qquad \text{and } K = K(u) + K(u_h),$$

in (21) we will have:

$$\|V-V_h\| \le K\ \Psi^m + \left(m\ L_k + (m + \frac{\Lambda}{\alpha})\ L_v \right) h$$

but the right hand side is valid $\forall$ m, so we can replace it by its minimum value $\bar{m} = \dfrac{1}{\ln \Psi} \ln \left[\dfrac{C_h\ h}{K} \right]$

$$\|V-V_h\| \le h\ C_h\ \ln \Psi + h \left(-C_h\ \ln \left[\frac{C_h\ h}{K} \right] + \frac{\Lambda}{\alpha} L_v \right)$$
$$\le h\ C_h \left(\ln \Psi - \ln \left[\frac{C_h\ h}{K} \right] + \frac{\Lambda}{\alpha C_h} L_v \right)$$

but, for $h \le \dfrac{\Psi\ K}{C_h}\ e^{\Lambda L_v / \alpha C_h}$ it holds

$$\|V-V_h\| \le C_h\ h\ \ln \left[\frac{\Psi\ K}{C_h}\ e^{\Lambda L_v / \alpha C_h}\ \ 1/h \right]$$

Hence, if we call

$$C_h = - \frac{L_k + L_v}{\ln \Psi} \qquad \text{and} \qquad D_h = \frac{\Psi\ K}{C_h}\ e^{\Lambda L_v / \alpha C_h}$$

we will get

$$\|V-V_h\| \le C_h\ h\ \ln \left[D_h / h \right]$$

4.3 Algorithm

Let $T_h : \mathcal{W}_h \longrightarrow \mathcal{W}_h$ be the following operator:

$$T_h w(x^k) = \min \begin{cases} \dfrac{1}{\alpha + \Lambda} \left\{ f(x^k) + \displaystyle\sum_{j=1}^{M} \lambda_j * \left[\sum_{\zeta \in \vartheta^+} w(x^k - \Delta\xi_{\zeta j} e_j)\, m_{\zeta j} + \right. \right. \\ \qquad \left. \left. + \displaystyle\sum_{\zeta \in \vartheta^-} \left[w(x^k + (\underline{x}_j - x^k_j) e_j) + \varphi_j(\underline{x}_j + \Delta\xi_{\zeta j} - x^k_j) \right] m_{\zeta j} \right] \right\} \\ \mathcal{M}_h(w)(x^k) \end{cases} \tag{23}$$

We apply a similar algorithm to that proposed in [5].

Algorithm 0:

Step 0: Give $w^{\circ} \in \mathcal{W}$ and set $m = 0$.

Step 1: Define $w^{m+1} = T_h(w^m)$

Step 2: Set $m = m+1$, and go to Step 1.

In the following theorem we stante some properties for the convergence of Algorithm 0:

Theorem 8:

i) T_h is a convex and increasing operator.

ii) $\forall \; w^{\circ} \in \mathscr{W}$, $\exists \; \underline{v}_h$ and $\bar{v}_h$ in $\mathscr{W}$ such that

$$\underline{v}_h \leq w^{\circ} \leq \bar{v}_h$$

and
$$T_h(\bar{v}_h) \leq \bar{v}_h$$

iii) $\exists \; \delta_h \in (0,1]$ such that $\quad \underline{v}_h + \delta_h(\bar{v}_h - \underline{v}_h) \leq T_h(\underline{v}_h)$

iv) Algorithm 0 converges to V_h, for any initial point $w^{\circ} \in \mathscr{W}$, moreover, it holds:

$$\| T_h^{\,m}(w^{\circ}) - V_h \| \leq K(w^{\circ}) \; (1-\delta_h(w^{\circ}))^m$$

Proof:

i) T_h is clearly an increasing function. Let us show it is also convex. We can rewrite T_h as:

$$(T_h(w))(x^k) = \min \left(\beta l_h(w,\phi,f), \; \mathcal{M}_h(w)(x^k)\right)$$

with $l_h(\cdot,\phi,f)$ is the affine function defined in (23). Let $u,v \in \mathscr{W}$ and $\theta \in [0,1]$. We may have:

- $T_h(u) \leq \beta l_h(u,\phi,f)$, and $T_h(v) \leq \beta l_h(v,\phi,f)$, and the convex sum will verify:

$$\theta \; T_h(u) + (1-\theta) \; T_h(v) \leq \theta \; \beta \; l_h(u,\phi,f) + (1-\theta) \; \beta \; l_h(v,\phi,f)$$

but operator $I_h(\cdot,\phi,f)$ is linear, so

$$\theta\, T_h(u) + (1-\theta)\, T_h(v) \le \beta\, I_h(\theta u+(1-\theta)v,\phi,f)$$

- $T_h(u) \le \mathcal{M}_h(u) \le k(q) + u(x^k+q) \quad \forall q$, and

  $T_h(v) \le \mathcal{M}_h(v) \le k(q) + v(x^k+q) \quad \forall q$. Therefore, the convex sum

will give us

$$\theta\, T_h(u) + (1-\theta)\, T_h(v) \le \theta\, (k(q) + u(x^k+q)) \;+$$

$$+ (1-\theta)\, (k(q) + v(x^k+q)) \le \mathcal{M}_h(\theta u + (1-\theta)v)$$

Consequently, we have T_h is a convex operator.

ii) Given $w^o \in \mathcal{W}_h$, let

$$K(w^o) = \max\left[\, \|w^o\|,\ \frac{\|f\|}{\alpha} + \frac{\Lambda}{\alpha}\,\|\phi\| + k_o\frac{(\alpha+\Lambda)}{\alpha}\right] \tag{24}$$

then, we define $\underline{v}_h \equiv -K$ and $\bar{v}_h = K$, clearly $\underline{v}_h$ and $\bar{v}_h \in \mathcal{W}_h$

and also $\underline{v}_h \le w^o \le \bar{v}_h$.

Besides, as it holds

$$\beta I_h(\bar{v}_h,\phi,f) \le K(w^o) = \bar{v}_h$$

and $$\mathcal{M}_h(\bar{v}_h) \ge k_o + K(w^o) = \bar{v}_h$$

we will obtain:

$$T_h(\bar{v}_h) = \min\,(\beta I_h(\bar{v}_h,\phi,f),\ \mathcal{M}_h(\bar{v}_h)) =$$

$$= \beta I_h(\bar{v}_h,\phi,f) \le K(w^o) = \bar{v}_h$$

iii) We define δ_h by

$$\delta_h = \frac{k_o}{2K(w^o)}$$

Let $u_\delta = \underline{v}_h + \delta_h(\bar{v}_h - \underline{v}_h) = -K + \delta_h 2K = -K + \frac{k_o}{2K} 2K = -K + k_o$

We know

$$\mathcal{M}_h(\underline{v}_h) \geq k_o + \underline{v}_h = k_o - K(w^o)$$

So, $$u_\delta \leq \mathcal{M}_h(\underline{v}_h)$$

Besides,

$$\beta I_h(\underline{v}_h, \phi, f) \geq -\beta\ K(w^o) - \beta\ \Lambda\ \|\phi\| - \beta\ \|f\|$$

Adding $K(w^o)$ at both members, we obtain

$$\beta I_h(\underline{v}_h, \phi, f) \geq -\ K(w^o) + (1-\beta\Lambda)\ K(w^o) - \beta\ \Lambda\ \|\phi\| - \beta\ \|f\|$$

but from (24) we know

$$K(w^o) \geq \frac{\|f\|}{\alpha} + \frac{\Lambda}{\alpha}\ \|\phi\| + k_o \frac{(\alpha+\Lambda)}{\alpha}$$

so, as $(1-\beta\Lambda) = \dfrac{\alpha}{(\alpha+\Lambda)}$,

$$\beta I_h(\underline{v}_h, \phi, f) \geq -\ K(w^o) + \frac{\alpha}{(\alpha+\Lambda)} \left[\frac{\|f\|}{\alpha} + \frac{\Lambda}{\alpha}\ \|\phi\| + k_o \frac{(\alpha+\Lambda)}{\alpha} \right] -$$

$$- \beta\ \Lambda\ \|\phi\| \quad \beta\ \|f\|$$

that is,

$$\beta I_h(\underline{v}_h, \phi, f) \geq -\ K(w^o) + k_o = u_\delta$$

Consequently,

$$u_\delta \le \min\,(\beta I_h(\underline{v}_h,\phi,f),\ \mathcal{M}_h(\underline{v}_h)) = T_h(\underline{v}_h)$$

iv) We are now under the same hypothesis as Theorem 4, so we can assure:

$$= \ \| T_h^{\,m}(w^\circ) - V_h \| \le K(w^\circ)\ (1-\delta_h(w^\circ))^m$$

$$\forall\ w^{o} \in \mathcal{W}_h.$$

∎

Generally we can observe Algorithm 0 converges dismally slowly when the contraction factor $h = \dfrac{\Lambda}{\alpha+\Lambda}$ is close to unity, so we have modified it in order to improve computational times.

We introduce the following operators, for

$$\mathfrak{y} \in \mathfrak{Y}(x^k) = \left\{ 0,\ 1,\ \ldots\ \mathfrak{J}(x^k) \right\},$$

where $\forall\ k\ \ \exists$ a bijection

$$\left\{ 0,\ 1,\ \ldots\ \mathfrak{J}(x^k) \right\} \underset{\mathfrak{B}^{-1}}{\overset{\mathfrak{B}}{\longleftrightarrow}} \left\{ (q_2,\ldots,q_N)\ /\ q_i \in Q_i(x^k)\ ,\ \text{with} \sum q_i \le a_I^k \right\} \cup \cup \left\{ q_1 \in Q_1(x^k) \right\}$$

and we will call $\mathfrak{Z}(\mathfrak{y}) = \sum q_i e_i - \sum_{i>1} q_i e_I$ the control associated to the corresponding image of $\mathfrak{y}$ by $\mathfrak{B}$.

- For $\mathfrak{y} = 0$, it is

$$(\mathfrak{L}_h^{\mathfrak{y}}(w))(x^k) = \frac{1}{\alpha + \Lambda} \left\{ f(x^k) + \sum_{j=1}^{N} \lambda_j * \left[\sum_{\jmath \in \vartheta^+} w(x^k - \Delta\xi_{\jmath j} e_j) \, m_{\jmath j} + \right.\right.$$

$$\left.\left. + \sum_{\jmath \in \vartheta^-} \left[w(x^k + (\underline{x}_j - x_j^k) e_j) + \varphi_j(\underline{x}_j - x_j^k + \Delta\xi_{\jmath j}) \right] m_{\jmath j} \right] \right\}$$

- For $\mathfrak{y} > 0$,

$$(\mathfrak{L}_h^{\mathfrak{y}}(w))(x^k) = k(\mathfrak{Z}(\mathfrak{y})) + w(x^k + \mathfrak{Z}(\mathfrak{y}))$$

Obviously,

$$(T_h(w))(x^k) = \min \left\{ (\mathfrak{L}_h^{\mathfrak{y}}(w))(x^k) \; / \; \mathfrak{y} \in \mathfrak{Y}(x^k) \right\}$$

With these notations we are able to introduce the numerical algorithm employed for computations.

Description of the algorithm.

Algorithm 1:

Step 0: Take $w^{oo} \in \mathbb{R}^{N_h}$, $p_{max} \geq 1$. Set $m = 0$, $\mu = 0$, and start th

procedure.

Step 1: Define $w^{m+1,\mu}(x^k) = (T_h(w^{m,\mu}))(x^k)$.

Determine $\bar{\eta}(m,k)$ such that

$$w^{m+1,\mu}(x^k) = (\mathcal{L}_h^{\bar{\eta}(m,k)}(w^{m,\mu}))(x^k) \quad .$$

Step 2: If $w^{m+1,\mu}(x^k) = w^{m+1,\mu}(x^k)$ $\forall$ k, then stop; else go to

Step 3.

Step 3: For $m \geq 1$, compute $q = \text{card}\left\{ k \;/\; \bar{\eta}(m,k) \neq \bar{\eta}(m-1,k) \right\}$.

Step 4: If $q = 0$ then set $p = p + 1$, else set $p = 0$.

Step 5: If $p \leq p_{max}$ then set $m = m + 1$ and go to Step 1; else go

to Step 6.

Step 6: Define $\hat{\eta}(\mu,k) = \bar{\eta}(m,k)$ and solve the system

$$y^{\mu}(x^k) = (\mathcal{L}_h^{\hat{\eta}(\mu,k)}(y^{\mu}))(x^k) \quad .$$

Step 7: Set $m = 0$, $\quad w^{o\;\mu+1}(x^k) = y^{\mu}(x^k)$, $\forall$ k;

$\mu = \mu + 1$ and go to Step 1.

Theorem 9:

Algorithm 1 converges after a finite number of iterations, $\forall\ w^{oo}$ initial point given.

The convergence of the new algorithm can be proved following essentially the proof presented in [5], and using the convergence stated above for Algorithm 0 as a fundamental tool.

We show now some comparisons between computing times of Algorithm 0 and Algorithm 1. We can observe the strong dependence of the acceleration phenomenum on the contraction factor $h = \frac{\Lambda}{\alpha + \Lambda}$. The proposed Algorithm 1 shows its efficiency specially when h is close to unity. The results shown have been produced in a VAX 720, for the solution of an optimal control problem posed for an arborescent multilevel system with N = 5 and N_h = 1024.

h	CPU time Alg. 0 (in sec.)	CPU time Alg. 1 (in sec.)	% Reduction
0.50	22.33	17.76	23.13
0.86	70.58	20.00	71.66
0.91	108.71	21.01	81.68
0.96	300.73	22.09	92.65
0.99	991.02	22.43	97.74

5. APPLICATION OF THE NUMERICAL METHOD TO AN EXAMPLE WITH EXACT ANALYTICAL SOLUTION

Here we compare the exact solution of a simple problem with different numerical solutions found by the discretization method presented above.

It is a unidimensional problem for which we have just one possible input of demand, of amount $D_1=1$, and probability $p_1=1$.

Stock varies between 0 and 4, there is no stocking cost ($f\equiv 0$), and the ordering cost $k(\cdot)$ is constant.

As there is only one possible variation for the amount of demand, we can write (7) as

$$V(x) = \min \begin{cases} \dfrac{\lambda}{\alpha+\lambda} \quad [V(P(x,1)+\Phi(x,1)] \\ \min \left[k + V\,(x+\nu)/\ x+\nu \in [0,4] \right] \end{cases}$$

where

$$P(x,1) = \begin{cases} x-1 & \text{if } x-1 \geq 0 \\ 0 & \text{if } x-1 < 0 \end{cases}$$

$$\Phi(x,1) = \begin{cases} 0 & \text{if } x-1 > 0 \\ \varphi(1-x) & \text{if } x-1 \leq 0 \end{cases}$$

For φ sufficiently large, at $x=0$ the system poses an order of the maximum amount, because k is constant and there is no stocking cost. Hence

$$V(0) = k + V(4) \tag{25}$$

Also, there must be a point $\xi < 1$ where the system orders up to 4, and after which the optimal policy is to let the system freely evolve. For this ξ, the transition point between policies, it holds

$$k + V(4) = V(\xi) = \frac{\lambda}{\alpha+\lambda} (V(P(\xi,1)) + \Phi(\xi,1))$$

and, as $\xi - 1 < 0$, it is

$$k + V(4) = \frac{\lambda}{\alpha+\lambda} \left[V(0) + \varphi(1-\xi) \right]$$

but (25) gives

$$k + V(4) = \frac{\lambda}{\alpha+\lambda} \left[k + V(4) + \varphi(1-\xi) \right]$$

$$\left[k + V(4) \right] . \left[1 - \frac{\lambda}{\alpha+\lambda} \right] = \frac{\lambda}{\alpha+\lambda} \varphi(1-\xi)$$

that is

$$\left[k + V(4) \right] . \alpha = \lambda \varphi(1-\xi) \tag{26}$$

At the upper bound ($x=4$), we can only have free evolution, so

$$V(4) = \frac{\lambda}{\alpha+\lambda} V(3)$$

recursively, since we only order at $\xi < 1$,

$$V(4) = \left[\frac{\lambda}{\alpha+\lambda} \right]^2 V(2)$$

....

$$V(4) = \left[\frac{\lambda}{\alpha+\lambda} \right]^4 V(0)$$

Again by (25),

$$V(4) = \left[\frac{\lambda}{\alpha+\lambda} \right]^4 (k + V(4))$$

So from (26) we obtain (defining $\eta = \frac{\lambda}{\alpha+\lambda}$)

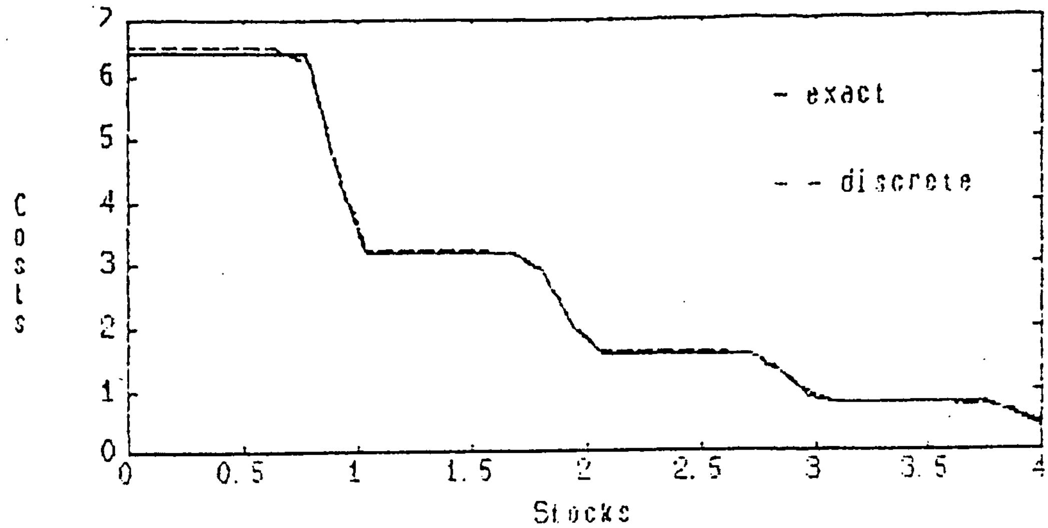
$$\xi = 1 - \frac{k \ \alpha}{\lambda \ \varphi \ (1-\eta)^4}$$

but we must have $\xi \geq 0$, so it must be verified

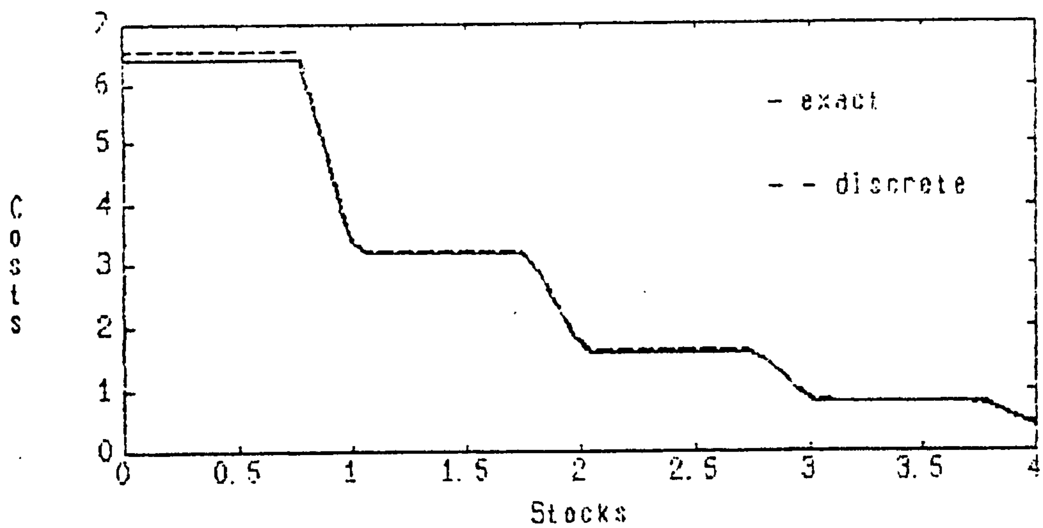
$$\frac{k \ \alpha}{\lambda \ \varphi \ (1-\eta)^4} \leq 1$$

For our numerical comparisons, we have put $\alpha = \lambda = 1$ (then $\eta = 0.5$), $\varphi = 27$ and $k = 6$. Then $\xi = 103/135$.

In Figures 2 and 3 are shown the exact solution and the approximated solution computed by our numerical procedure as well as differences between both functions for the example presented, but with two different discretisation steps (h=4/33 and h=4/55). Results have been obtained on a PC AT computer.



7
6
5
4
3
2
1
0
Costs
0
0.5
1
1.5
2
2.5
3
3.5
4
Stocks
- exact
- - discrete



7
6
5
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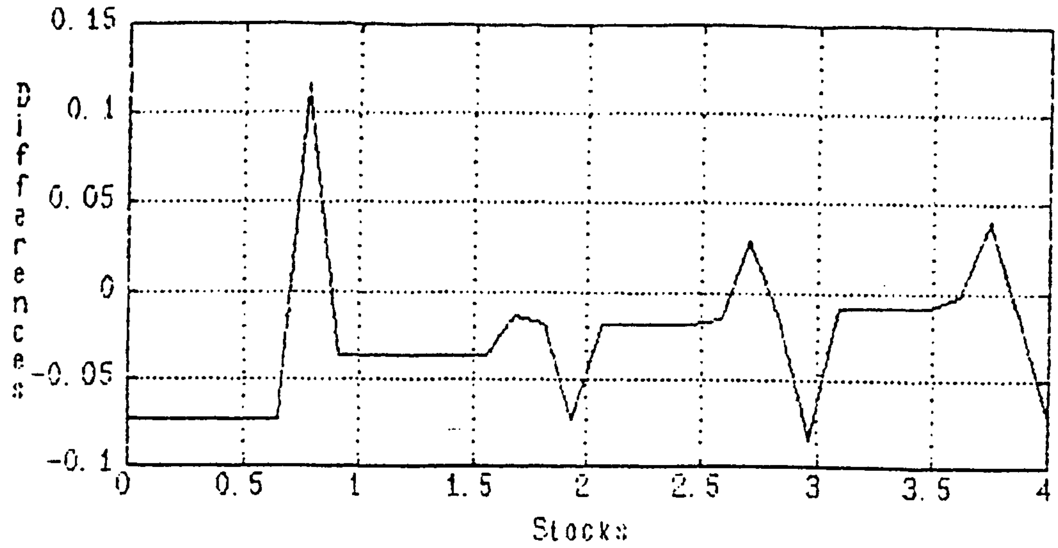
3
2
1
0
Costs
0
0.5
1
1.5
2
2.5
3
3.5
4
Stocks
- exact
- - discrete

Fig. 2


0. 15
0. 1
0. 05
0
-0. 05
-0. 1
Differences
0
0. 5
1
1. 5
2
2. 5
3
3. 5
4
Stocks

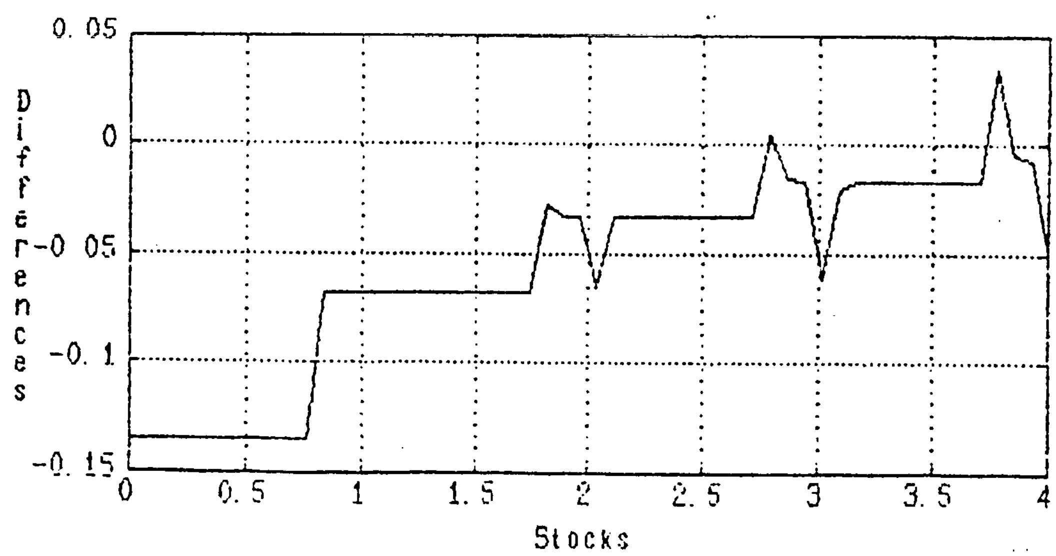

0. 05
0
-0 05
-0. 1
-0. 15
Differences
0
0. 5
1
1. 5
2
2. 5
3
3. 5
4
Stocks

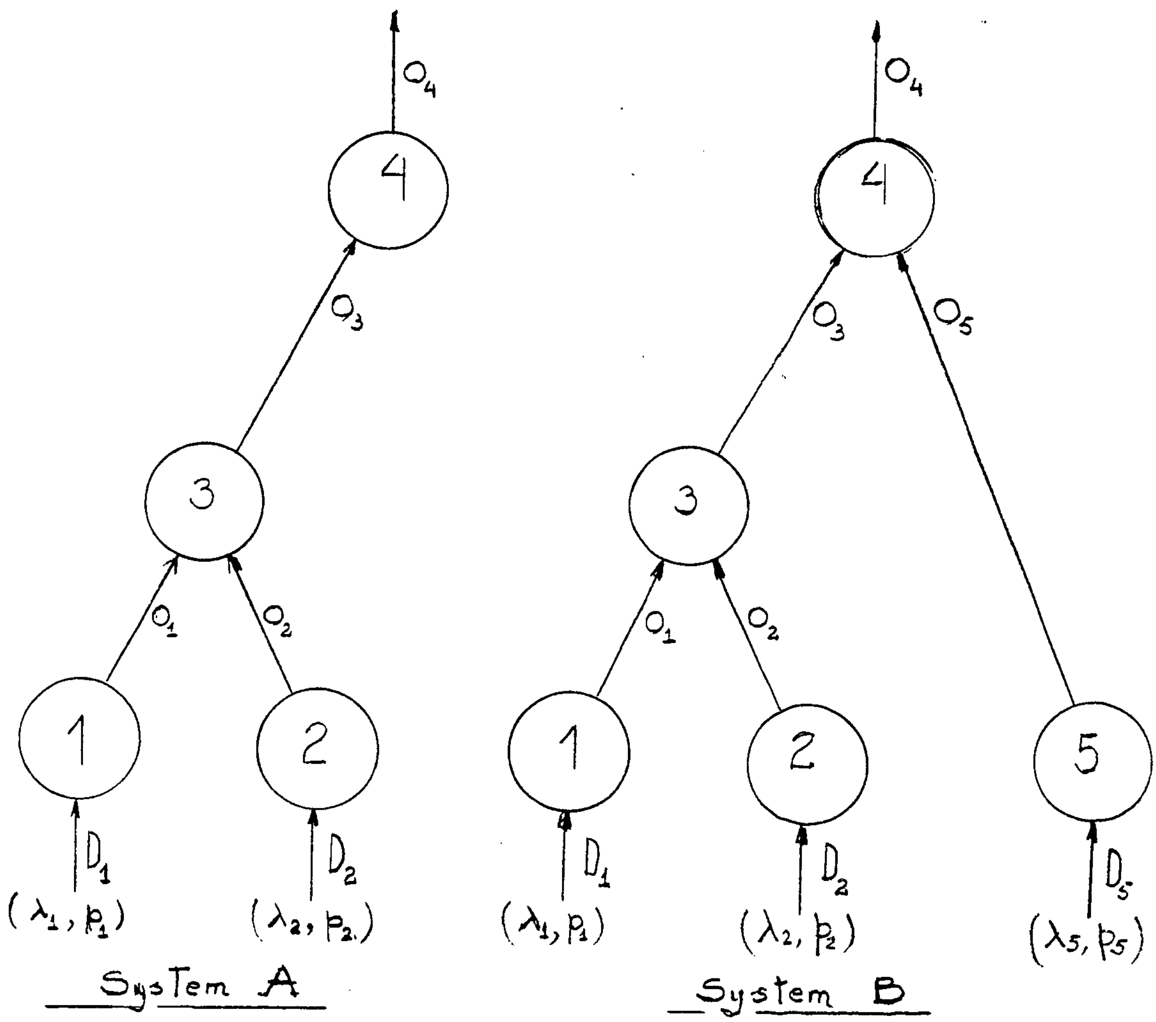
Fig. 3

6. COMPARISON OF TWO SYSTEMS WITH THREE HIERARCHICAL LEVELS

6.1. Description of the examples.

In this section we present some results obtained with the methodology proposed, considering two different possibilities for the number of installations. We show evolution of stocks, orders placed by each installation as well as the demands received under simulation of the system.

We have optimized the following systems:



where $\lambda_1=1$, $\lambda_2=1$, $\lambda_5=2$ represent the rate of arriving demands in each installation that is receiving outer demand, so the probability of one arrival at node i will be approximately given by $\lambda_i \Delta t$. Demand distributions are independent for each node. Another property of the model consists in allowing the input of different amounts of demand, these amounts are given by a random variable taking the following values:

$D_1 = 1.1$ (with probability $p_1=0.9$)

$D_2 = 1.7$ (with probability $p_2=0.1$)

This distribution has been taken identical in nodes 1, 2 and 5.

Each node has a maximum and a minimum stock, the last one may be negative if we consider backlogging (in that case we consider a maximum backlogging $|\underline{x}_i|$). We have discretized each continuous interval of stock $[\underline{x}_i,\bar{x}_i]$ into a set of N_i points. Each node can put an order of an arbitrary amount, provided it can be stocked and supplied. Consequently the continuous interval of orders $[0,\bar{x}_i-\underline{x}_i]$ is discretized into a set of N_{q_i} points. In particular, these values are:

Node	$\underline{x}_i$	$\bar{x}_i$	N_i	$N_{q_{i-}}$
1	-1	3	5	5
2	-1	3	5	5
3	0	10	5	5
4	0	60	3	3
5	-1	9	5	5

It is clear nodes 1,2,3 and 4 are the same in both systems. In fact, both systems are identical, except for the addition of an "extra" node at System B: installation 5.

The ordering cost has the following expression:

$$k(q) = \sum_i k_{oi} + k_{1i} \cdot q_i$$

For our examples we have:

Node	k_{oi}	k_{1i}
1	6.0	0.08
2	0.6	0.08
3	10	0.01
4	60	0.005
5	3	0.04

Finally, the stocking cost f has an additive structure. Each installation varies its cost (taken linear) according to the following criterium:

If $S \in [0,\bar{x}_i]$, then $f_i(S) = f_i^+ . S$ represents the real stocking cost.

If $S \in [\underline{x}_i,0)$, then $f_i(S) = f_i^- . S$ measures the cost related to the backlogging phenomenum.

When the entered demand ("D") is so big that it would make the current stock x_i go below $\underline{x}_i$, then this rupture of the maximum backlogging has to be paid according to the cost

$$|\varphi_i \cdot (x_i - D - \underline{x}_i)|$$

and the system stays at $\underline{x}_i$ (the demand accepted will just be $x_i - \underline{x}_i$).

For our examples, we have set

Node	f_i^+	$\lvert \bar{f}_i \rvert$	φ_i
1	0.1	80	45
2	0.1	80	45
3	0.007	———	———
4	0.0008	———	———
5	0.1	80	45

Remark: 3 and 4 do not have values for negative stocks, since they do not operate with backlogging.

Figures 5 to 9 show, by simulation of the system, evolution of stocks, orders placed by each installation as well as the demands received. Our aim being to compare numerical results obtained for Systems A and B, we have considered demands D_1 and D_2 are the same for both examples (see Figure 4).

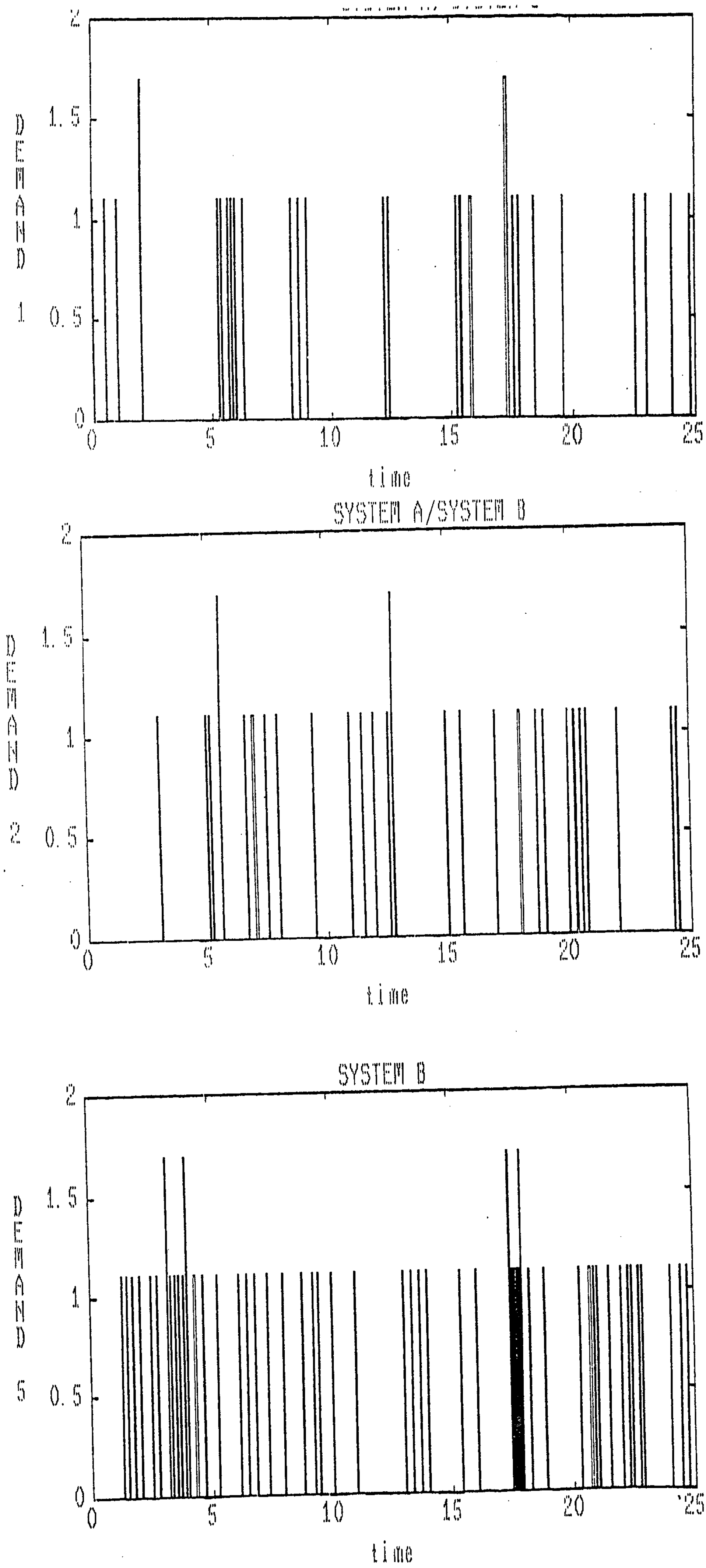
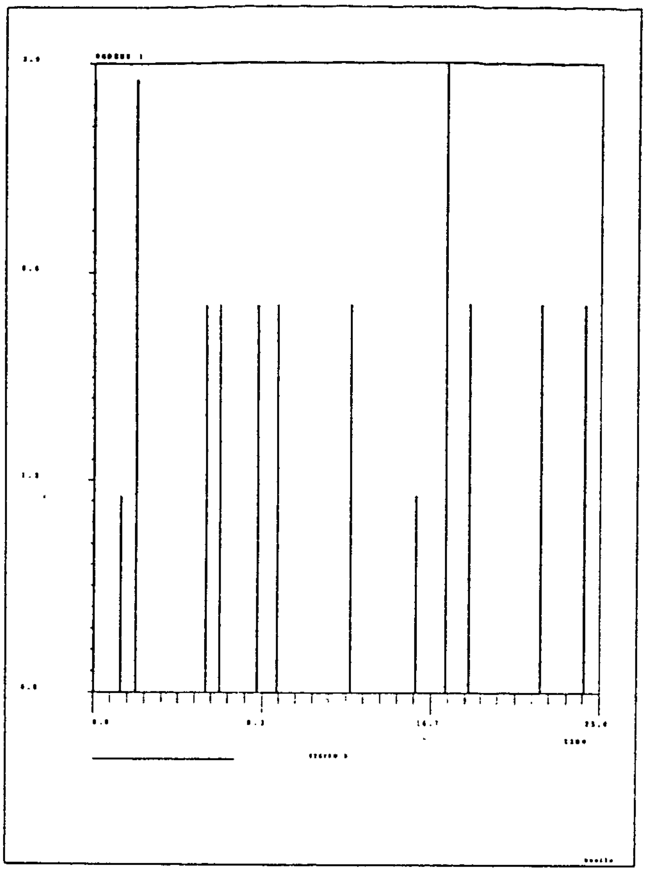
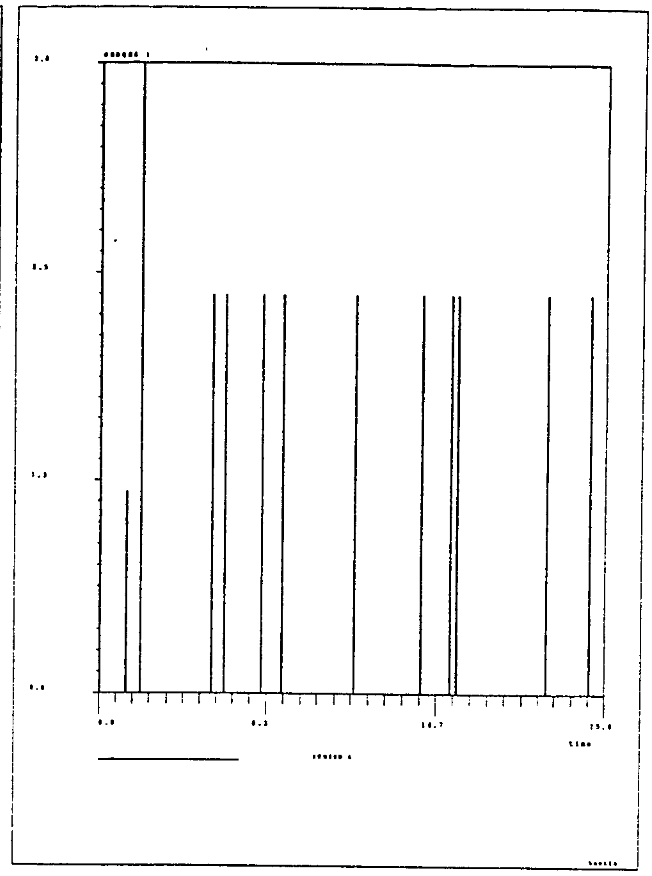

DEMAND 1
time
SYSTEM A/SYSTEM B
DEMAND 2
time
SYSTEM B
DEMAND 5
time

Fig. 4

ORDERS 1

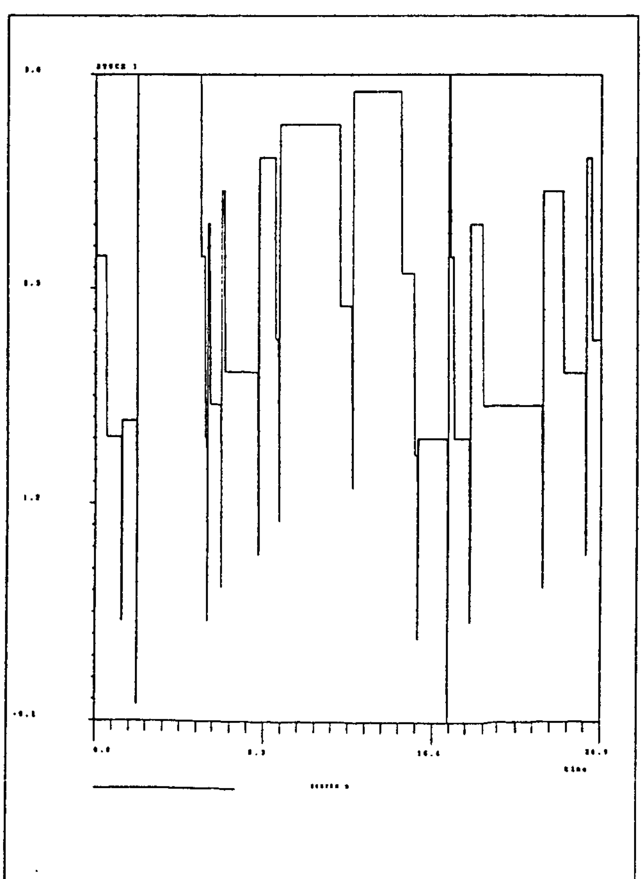




SYSTEM B

SYSTEM A

STOCK 1



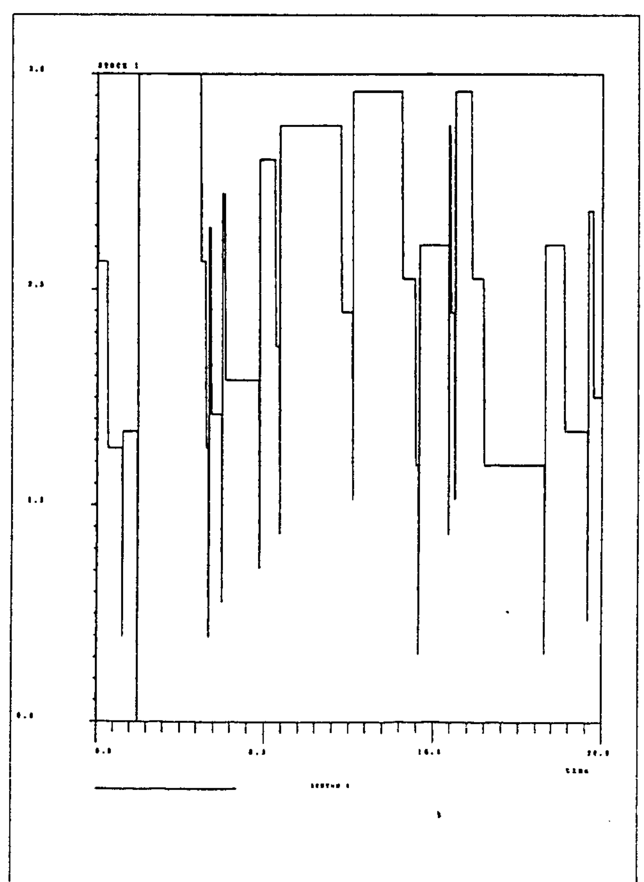
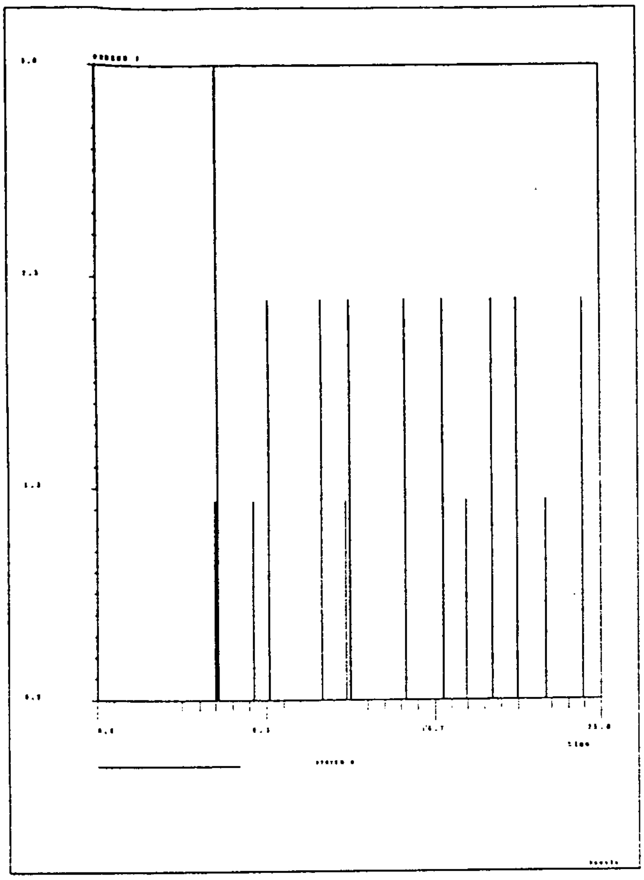
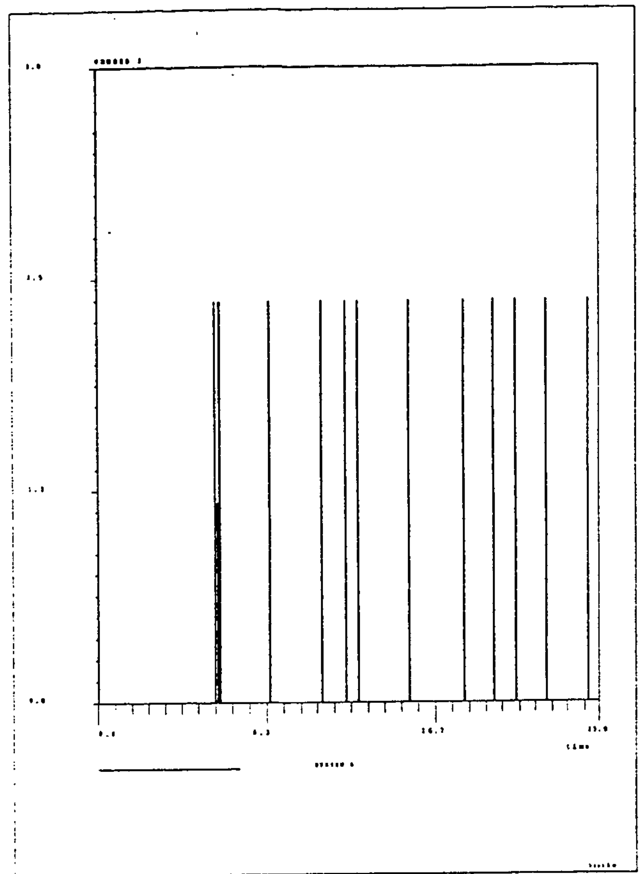


Fig 5

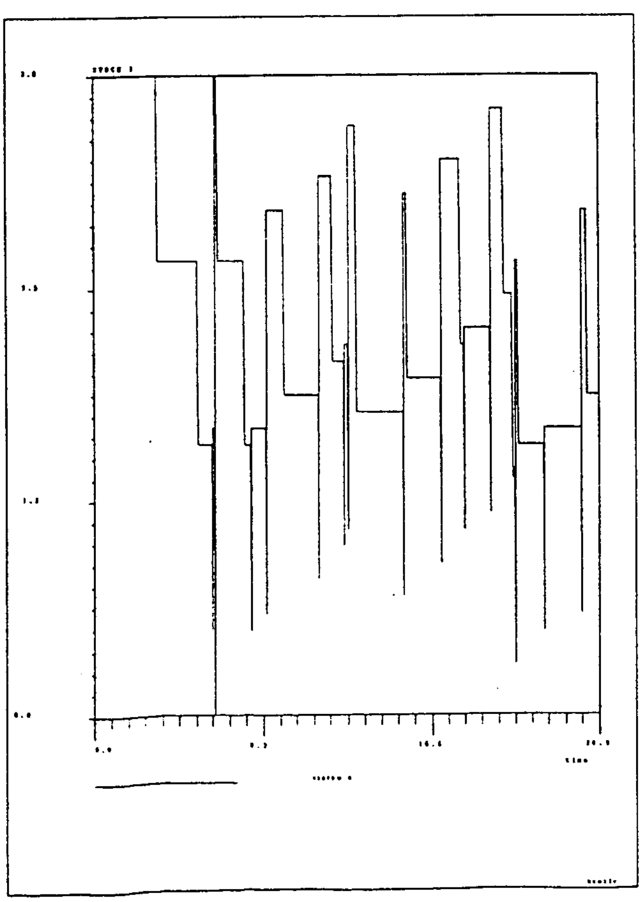


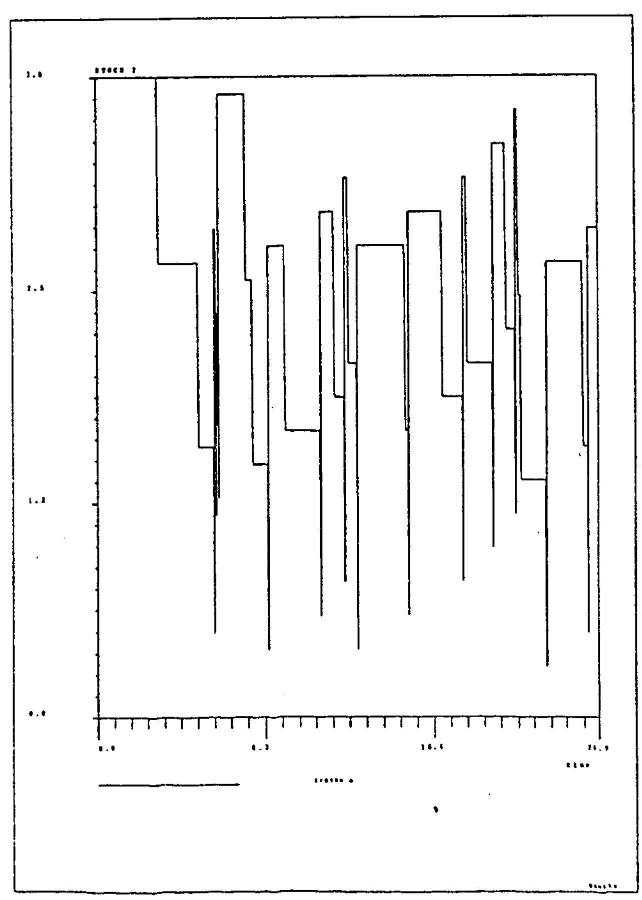


SYSTEM B

SYSTEM A

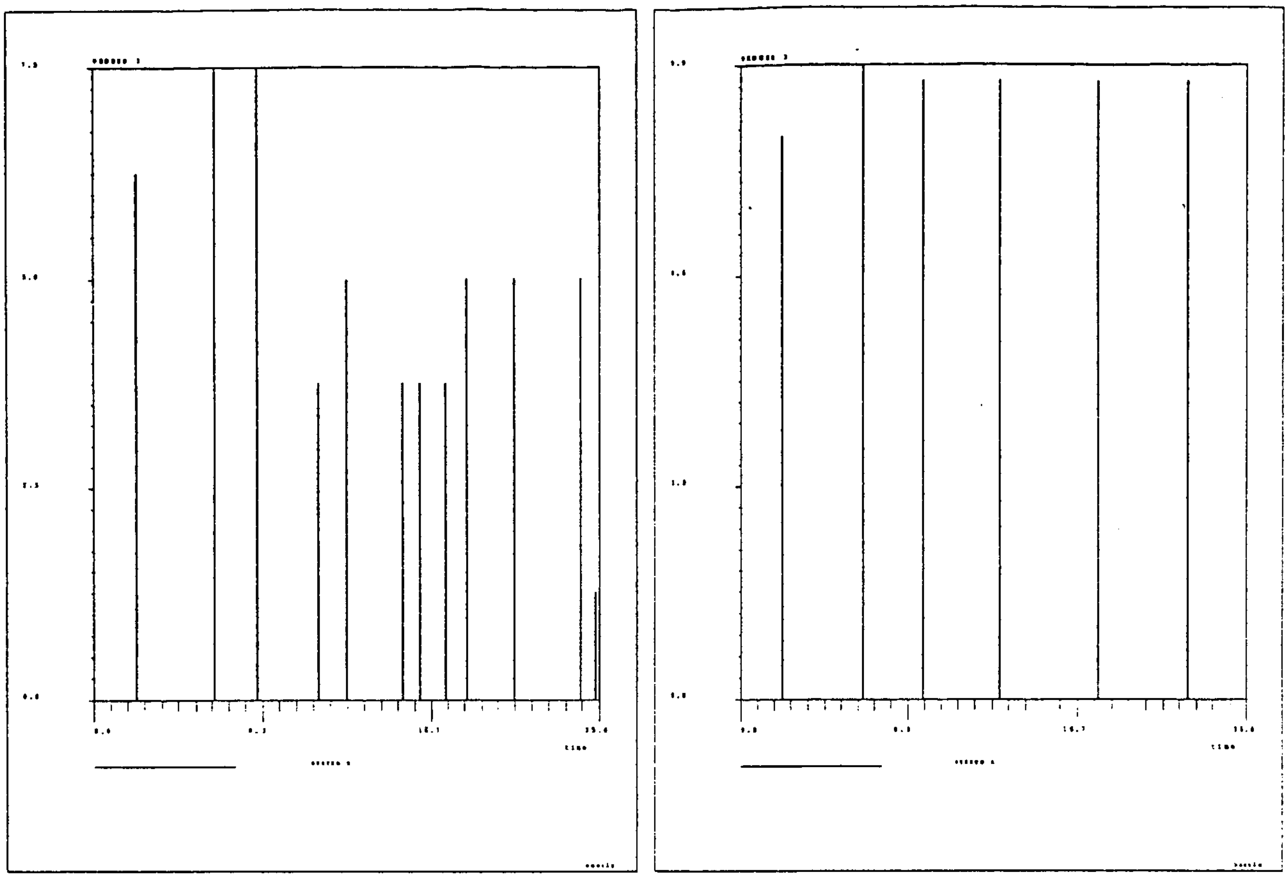
ORDERS 2





STOCK 2

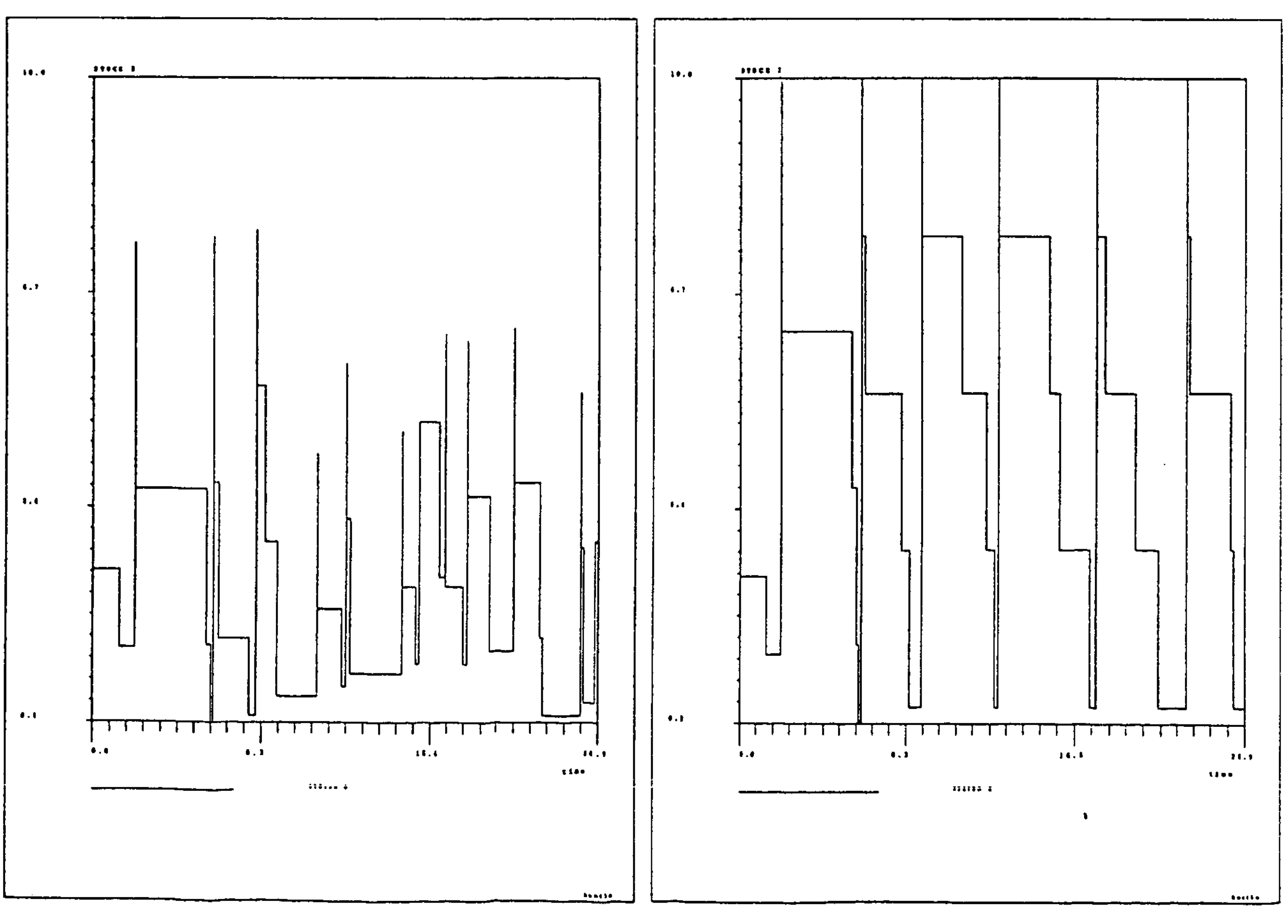
Fig 6



SYSTEM B

SYSTEM A

ORDERS 3



STOCK 3

Fig 7

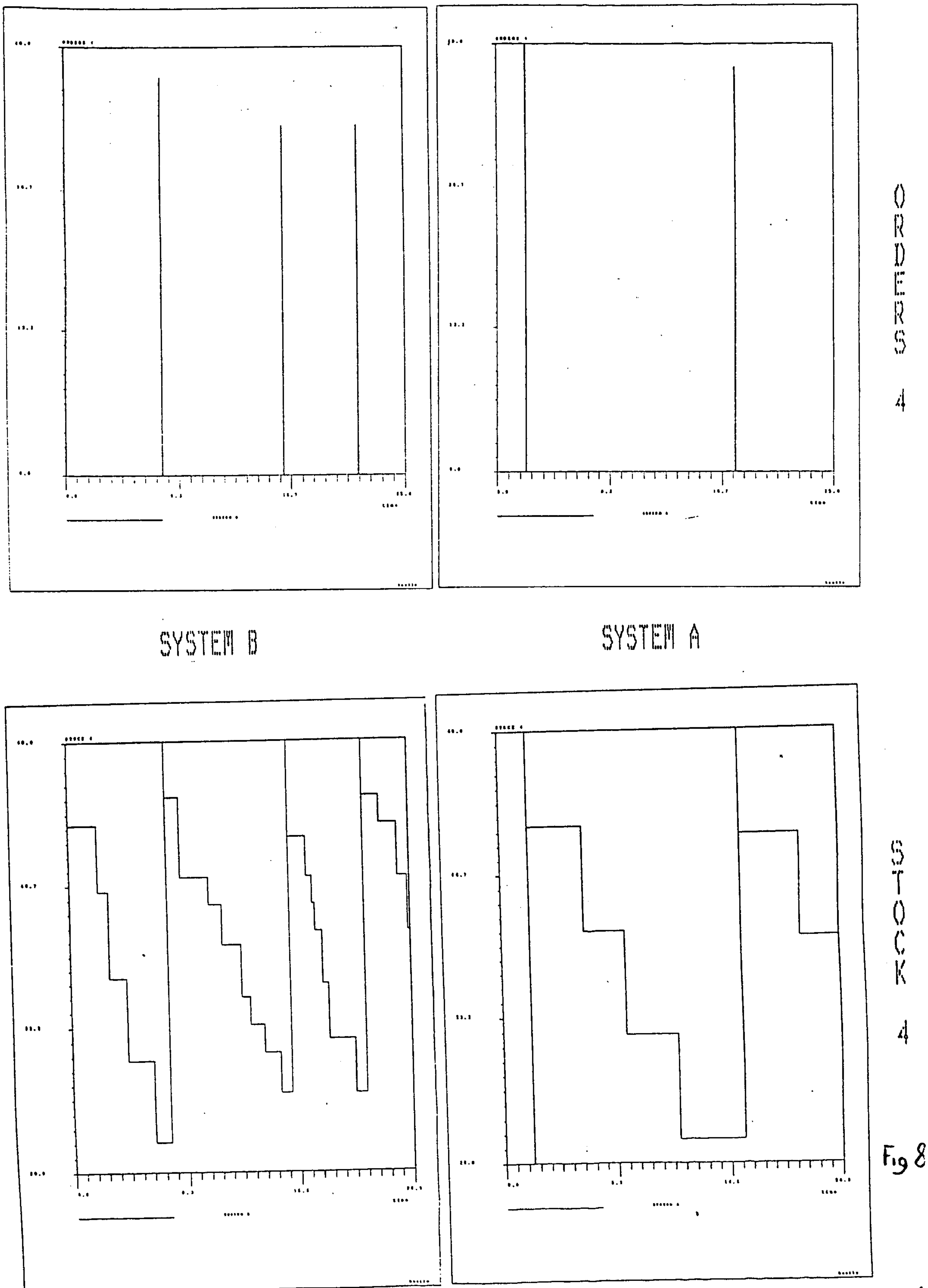
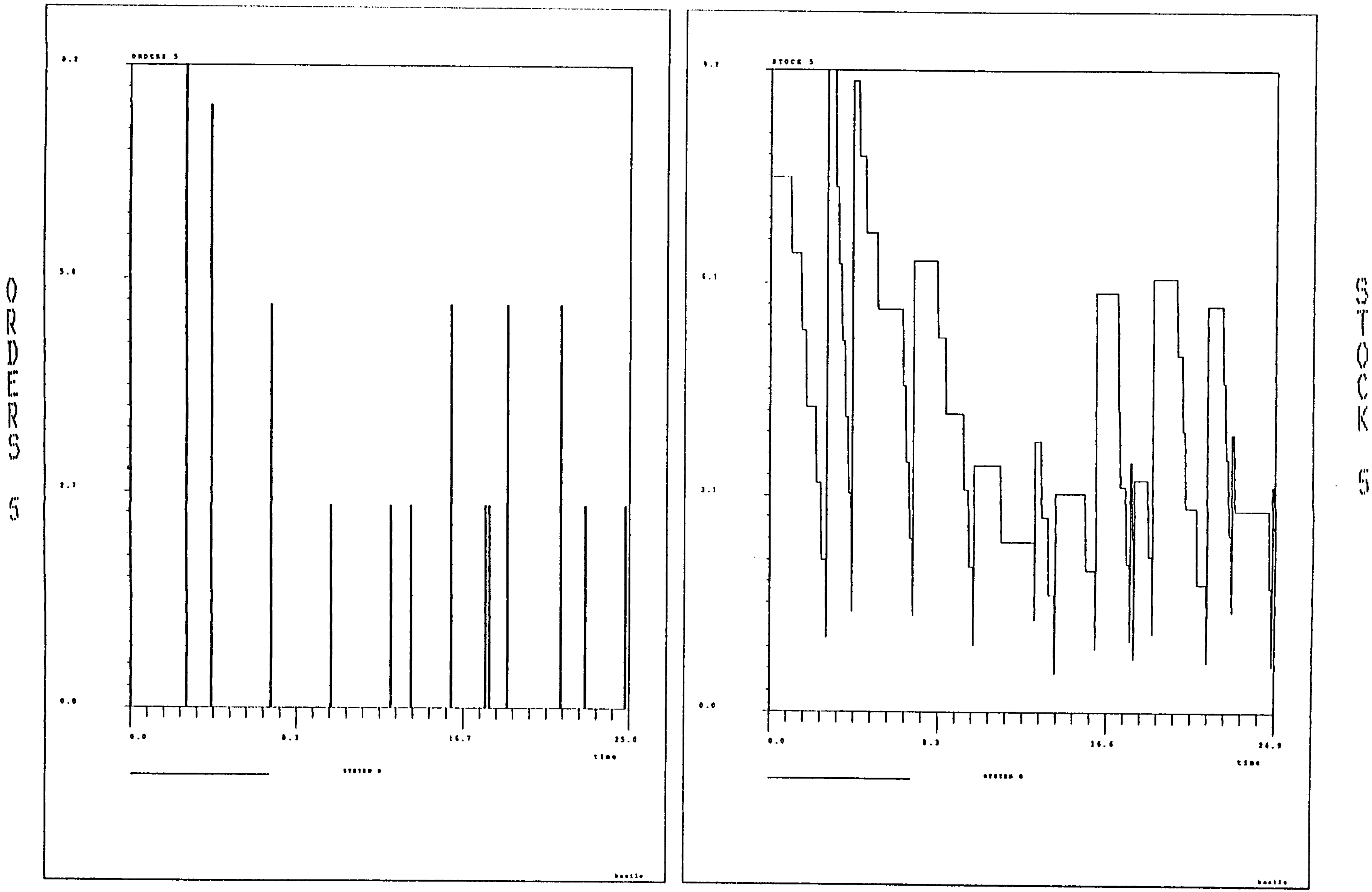


Fig 8

Fig. 9

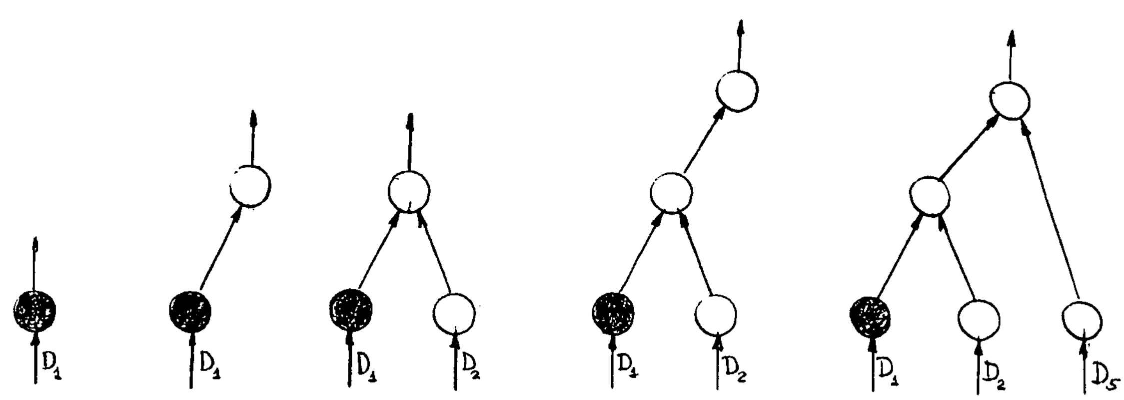
SYSTEM B



6.2. Some remarks concerning cooperation

6.2.1 Node 1 behaviour

For our comparisons we have performed global optimization for five different systems, all of then having installation 1 as a basic node. So we have started with a one installation system (namely node 1) and we have gone on adding a new node at each time (see Figure 10). Clearly, each centralized optimal policy suppose a different strategy for node 1. Table in Figure 10 shows how the addition of a new node makes node 1 adopt policies involving individual higher costs. Such augmentations represent cooperation node 1 is offering to the system in order to achieve an optimal global cost. Let us point out that a decentralized approach would allow node 1 keeping the lowest cost (22.4). In section 6.2.3 we will show the negative consequences of such a selfish action.



Operation Cost of NODE 1:	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5
	22.44	22.48	22.66	24.96	27.83
	-	0.2%	1%	11%	24%

Fig 10

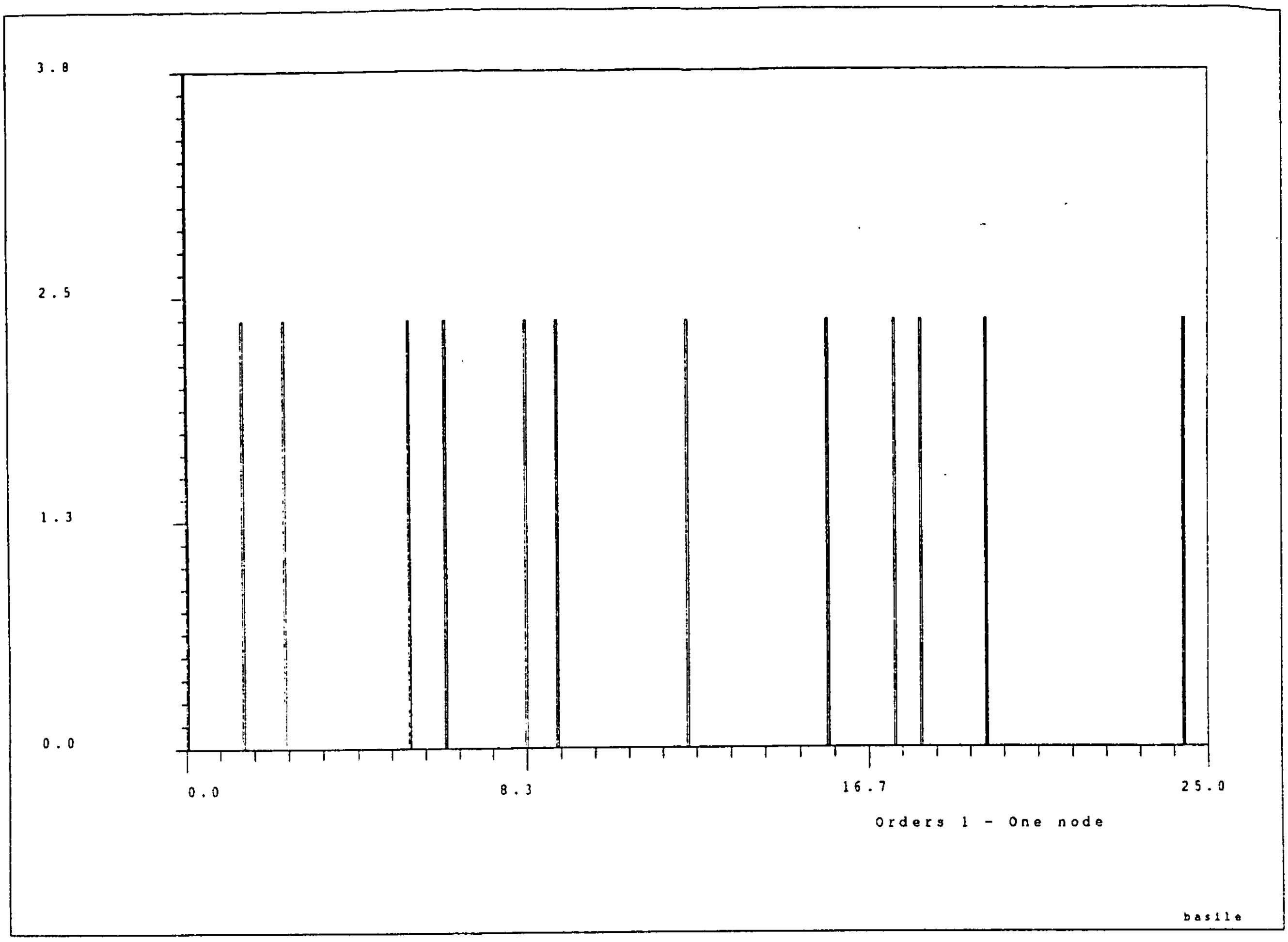
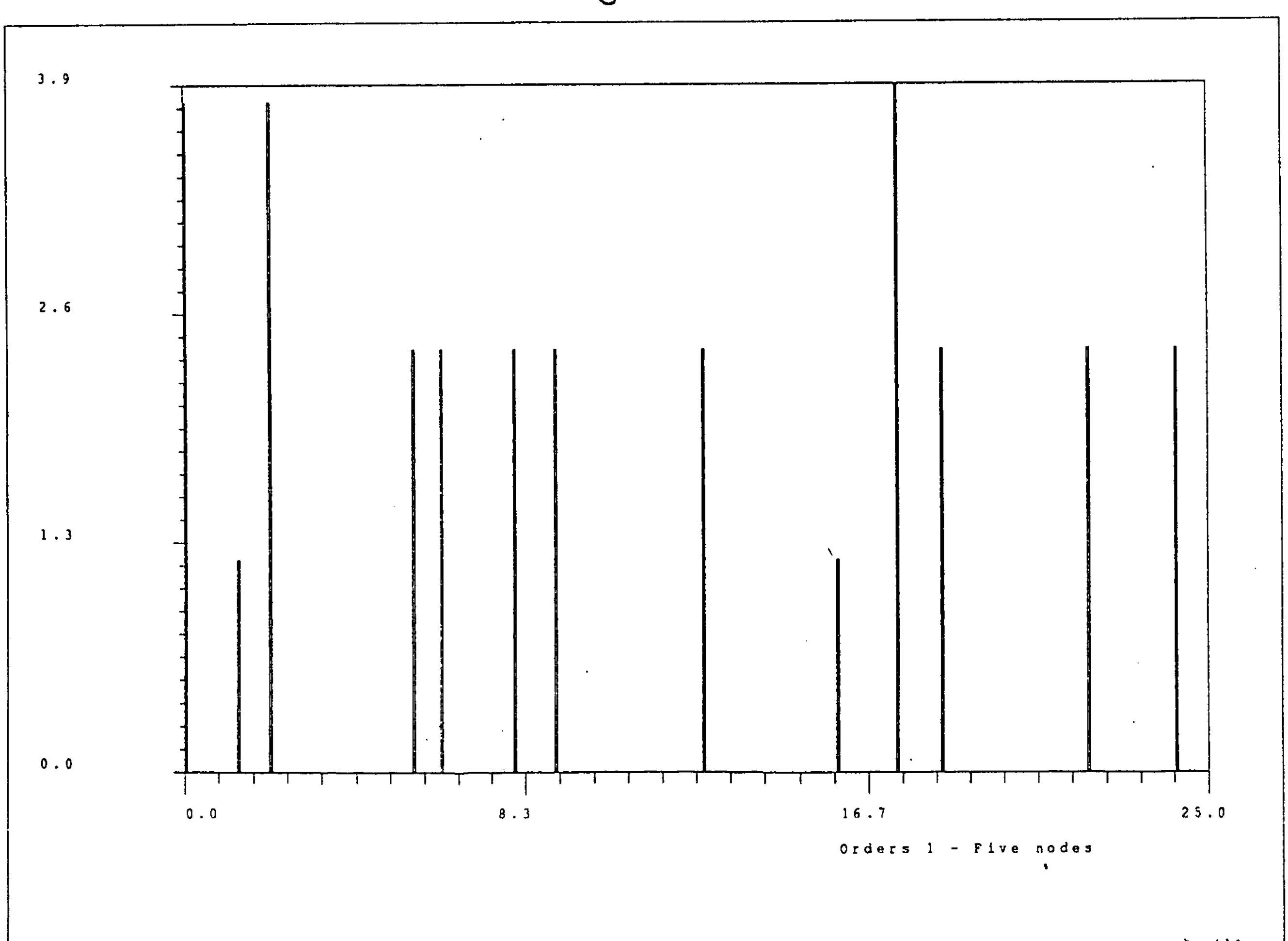
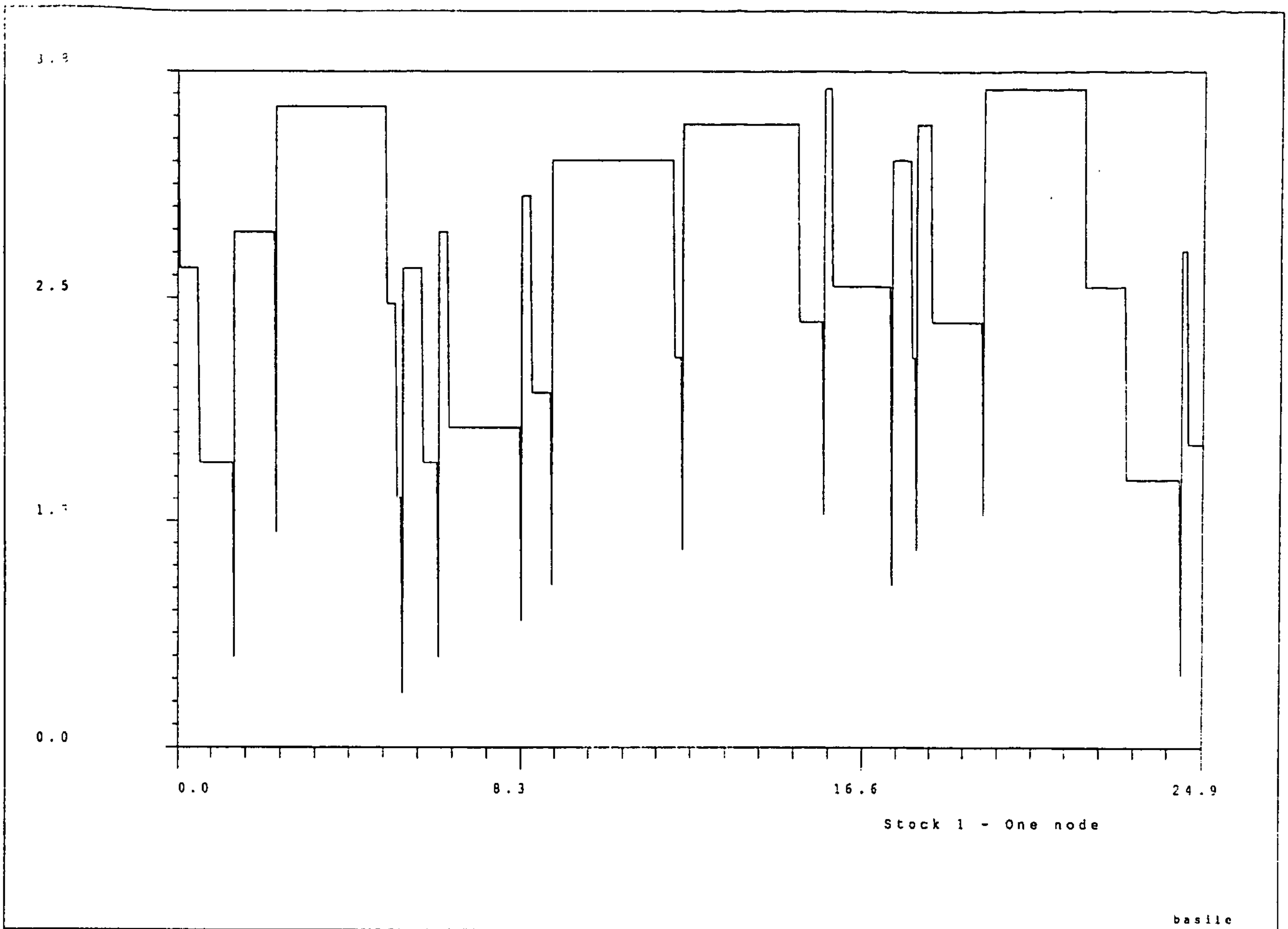

3.8
2.5
1.3
0.0
0.0
8.3
16.7
25.0
Orders 1 - One node
basile

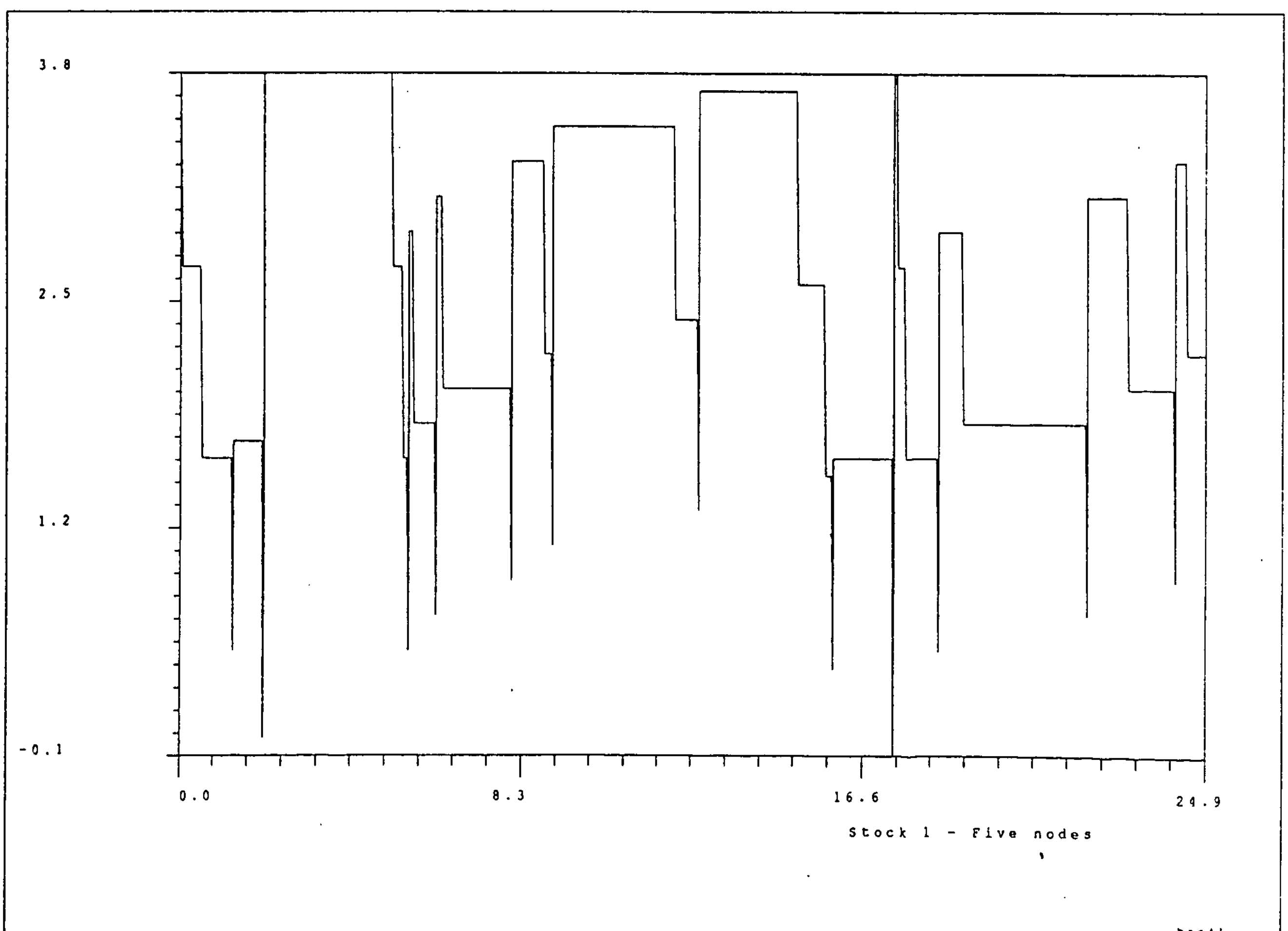
Fig 11


3.9
2.6
1.3
0.0
0.0
8.3
16.7
25.0
Orders 1 - Five nodes
basile



3.8
2.5
1.7
0.0
0.0
8.3
16.6
24.9
Stock 1 - One node
basile

Fig 12



3.8
2.5
1.2
-0.1
0.0
8.3
16.6
24.9
Stock 1 - Five nodes
basile

In Figures 11 and 12 we also show node 1 optimal orders and stock evolution for examples 1 and 5.

6.2.2. Subsystem (1-2-3)

As we have done in the preceeding section, Figure 13 shows increasing costs of subsystem (1-2-3) due to the succesive introduction of installations 4 (i.e. System A) and 5 (System B).

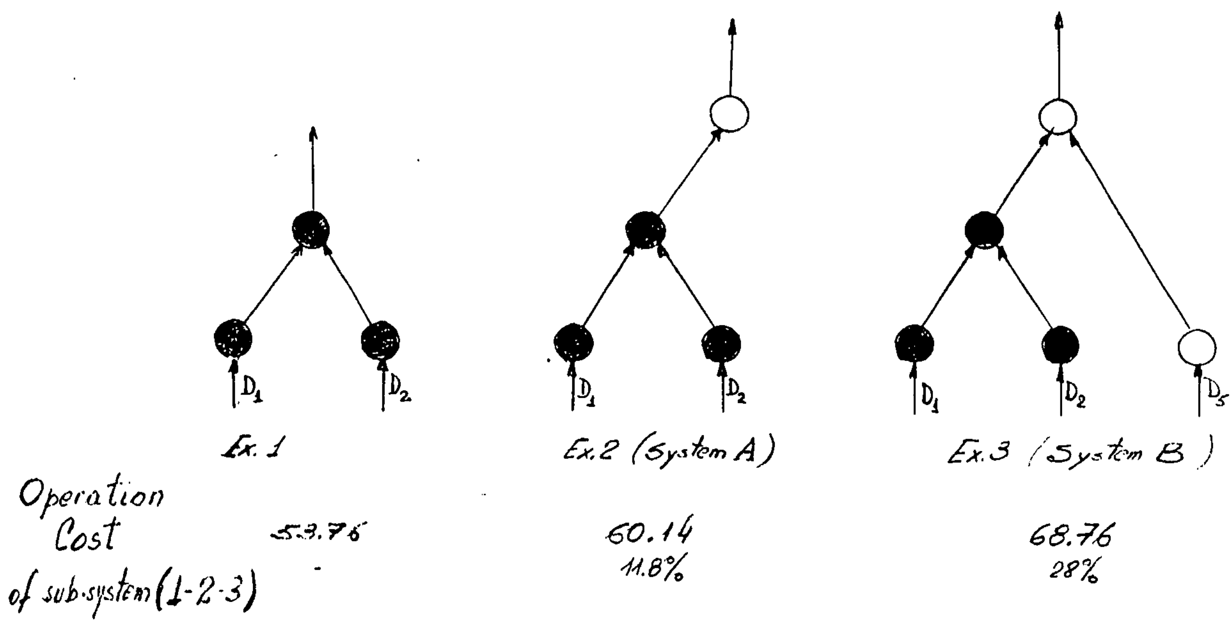


Fig. 13

6.2.3. Subsystems "selfishness" versus Cooperation

Let us consider subsystem (1-2-3). We already know that centralized optimization produces an optimal global cost of 139.99 for System B (see end of 6.1.). This centralized strategy suppose a cost equal to 68.76 for subsystem (1-2-3) (see Figure 13).

Assume now subsystem (1-2-3) imposes a condition for the entrance of node 5 to the system, since such incorporation would involve a higher cost for (1-2-3). This condition could be to give (1-2-3) a "priority" for being supplied. In this way its privilege over node 5 would let (1-2-3) get at least a cost closer to its former one (in System A, i.e. 60.14), although belonging now to System B. Table 1 shows how (1-2-3) gets decreasing costs as it asks for more and more privileges. But on the other side these "priorities" make System B increase its global cost with an amount much more significant than the reduction obtained by (1-2-3).

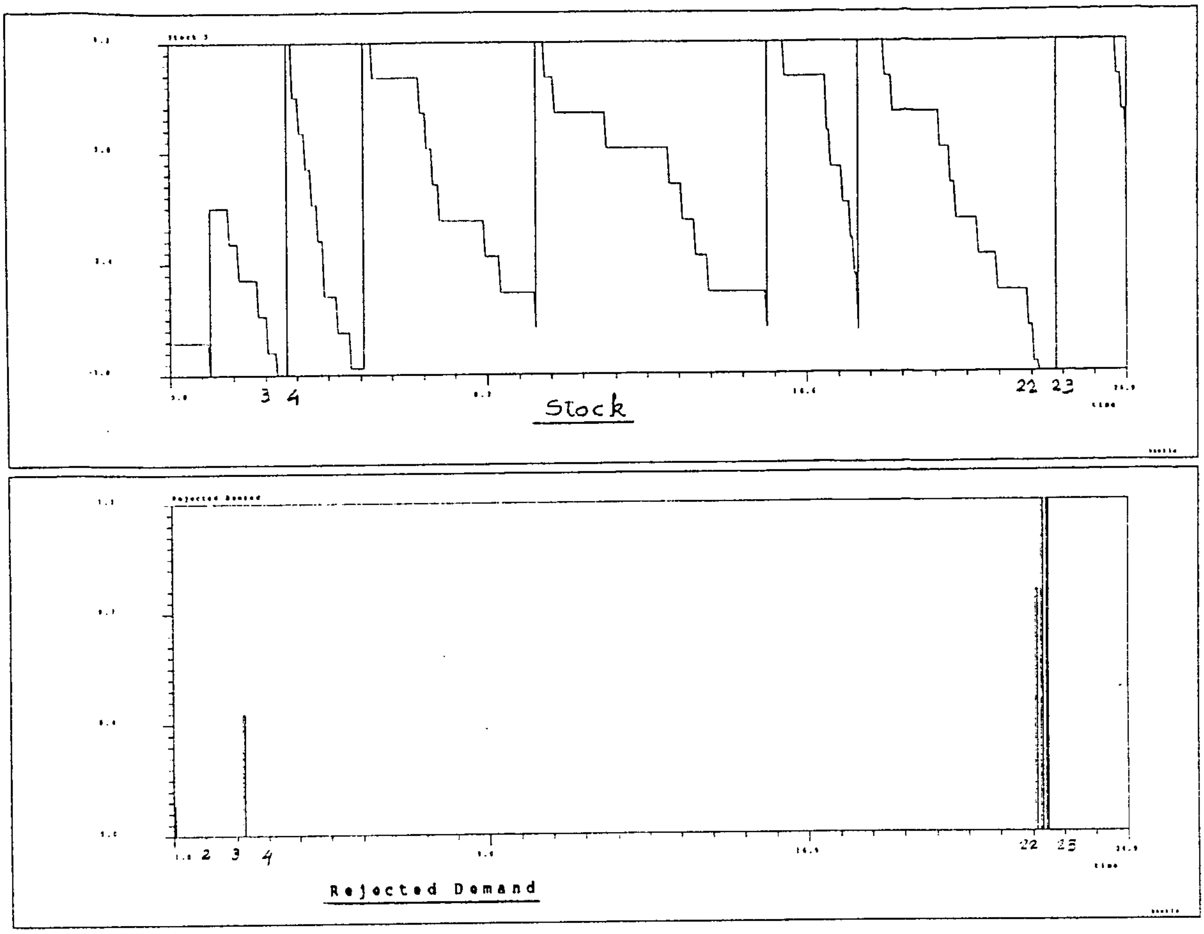
	Operation Cost of Subsystem (1-2-3) in B	Operation Cost of System B
(no privileges)	68.76	139.09
(*)	62.50	156.61
	60.20	160.28
	59.60	162.30
(**)	59.40	162.68

Table 1

We remark that System B increasing costs are due to the demand rejected by node 5 because of (1-2-3) supplying privileges (see Figures 14 and 15).

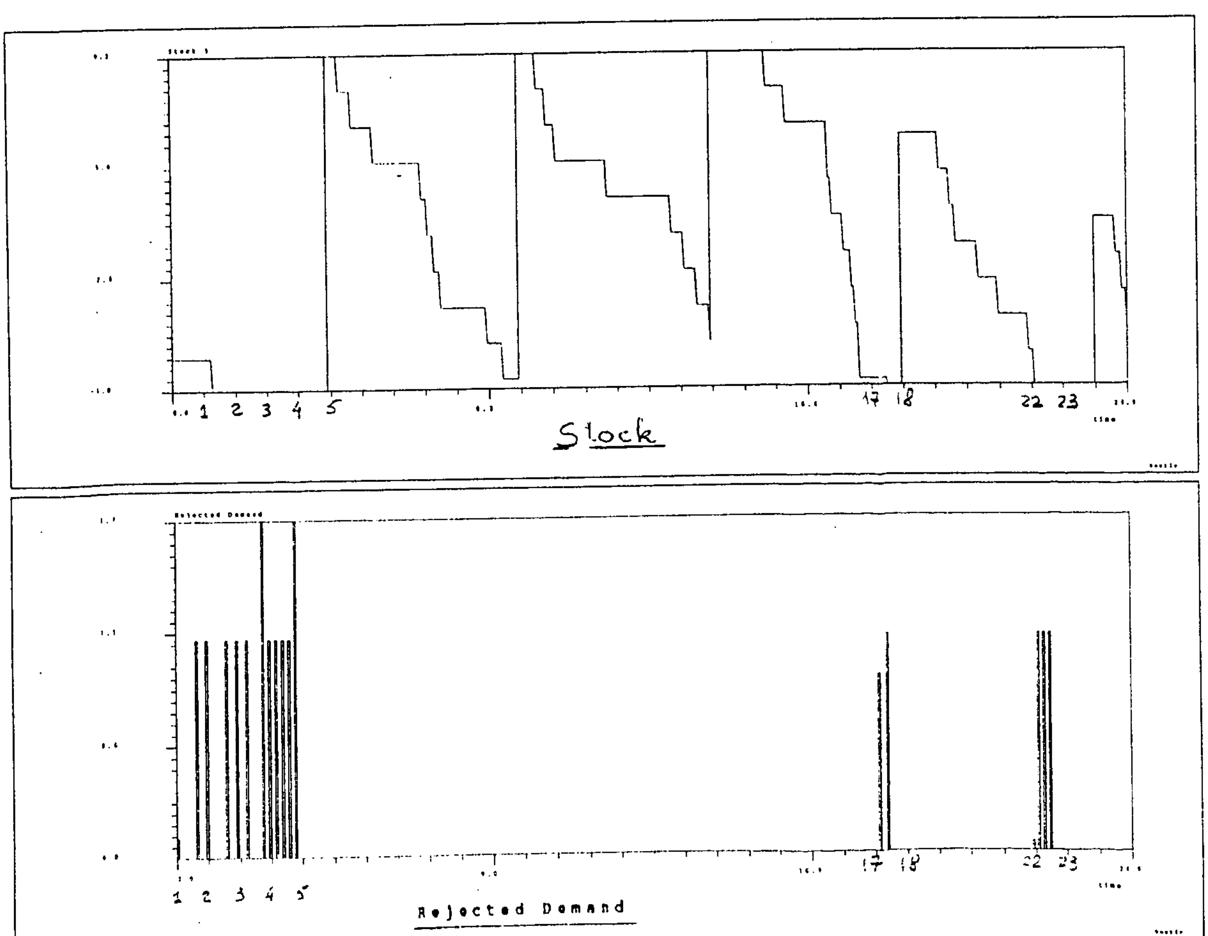
Summing up: if (1-2-3) intends to conserve its former cost (the one achieved in System A, 60.14) then System B has to increase its global cost in a 15%.

Node 5 face to selfish behaviour of subsystem (1-2-3)



Case (*)

Fig 14



Case (**)

Fig 15

Suppose now (1-2-3) refuses node 5 incorporation. We have as global cost of System A the addition of (1-2-3) cost and node 4 cost (60.14 and 18.58 respectively). In order to let node 5 operate, it should put its orders to a new "extra" node 4*, with identical characteristics as node 4. We would get then an auxiliary system (4*-5) which operates under global optimization with a cost of 29.37 for node 5 and 38.13 for 4*. Figure 16 shows the pernicious effect of considering two separated systems: System A and system (4*-5). Total costs are higher than System B global cost, although (1-2-3) and 5 obtain both better costs when operating separatedly.

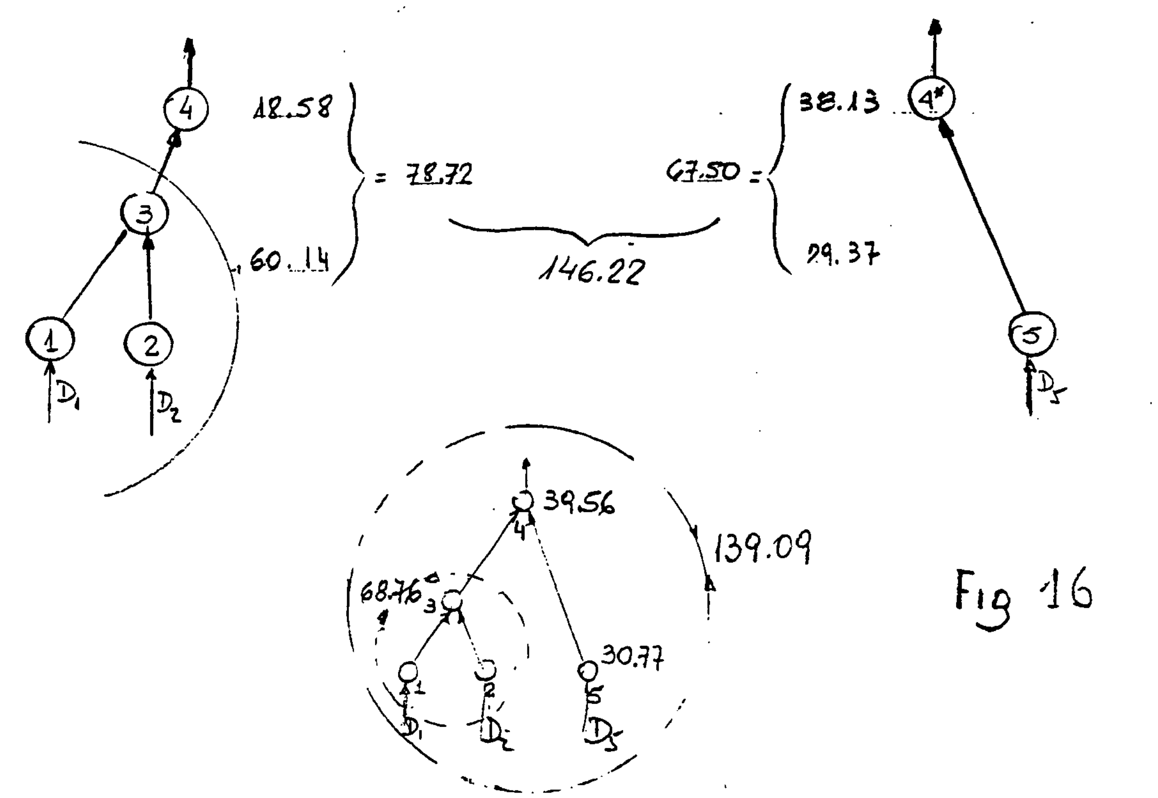


Fig 16

Summing up: A centralized optimization (System B operating costs) asks from:

- Subsystem (1-2-3) to *pay* a cooperative cost of

$$68.76 - 60.14 = 8.62$$

and from

- Node 5 to *pay* a cooperative cost of

$$30.77 - 29.37 = \underline{1.44}$$

Total cooperative costs = 10.02

But this cooperation allows the existence of just one node "4" at the maximum level of hierarchy. This fact reduces global cost in

$$(18.58 + 38.13) - 39.56 = 17.13$$

Hence in System B we save

$$17.13 - 10.02 = 7.11$$

That is, we get a reduction of approximatedly a 5% over the cost obtained when operating in a decentralized way (System A + (4*-5)).

7. CONCLUSIONS

We have presented a simple model for AMS that makes it possible to take capacity constraints at each installation into account. Some numerical examples have been worked through. Simulations of the system operation under approximate optimal policies show the usefulness of the proposed numerical method as well as the advantages of operating under global optimization.-

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